

On the Ramanujan differences

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Abstract. We consider functional equation

$$f(a+b+c) + f(b+c+d) + f(a-d) = f(a+b+d) + f(a+c+d) + f(b-c),$$

where $f : R \rightarrow G$, $a, b, c, d \in R$ satisfying $ad = bc$. The solutions are known for $G = R = \mathbb{R}$ and $R = \mathbb{Z}$, $G = \mathbb{R}$. The main result of the paper (Theorem 7) determines the solutions in the case when R is a field of characteristic zero and G is a linear space over $\mathbb{Q}$. Then $f : R \rightarrow G$ satisfies the equation if and only if there are additive functions $a_1, a_2 : R \rightarrow G$ and $a_0 \in G$ such that $f(x) = a_2(x^4) + a_1(x^2) + a_0$ for all $x \in R$.

1. Introduction

Let $R(+, \cdot)$ be a commutative ring with identity. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, R)$$

then let

$$\det A := ad - bc, \quad A^o := \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

Denote by $\text{Mat}^*(2, R)$ the set of the matrices $A \in \text{Mat}(2, R)$ for which $\det A = 0$. Let $G(+)$ be an Abelian group. If $f : R \rightarrow G$ is a function then let

$$(1) \quad C_f(A) := f(a+b+c) + f(b+c+d) + f(a-d)$$

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for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, R)$. Obviously, $C_f : \text{Mat}(2, R) \rightarrow G$. In this case the Ramanujan difference of the generating function $f : R \rightarrow G$ is defined by the equation

$$D_f(A) := C_f(A) - C_f(A^o)$$

for any $A \in \text{Mat}(2, R)$ (DARÓCZY [3]). Denote by $S(R, G)$ the set of all the functions $f : R \rightarrow G$ for which the functional equation

$$(2) \quad D_f(A) = 0 \quad \text{if } A \in \text{Mat}^*(2, R)$$

holds. The functional equation (2) means the following in detail. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad ad = bc$$

$(a, b, c, d \in R)$ then

$$(3) \quad \begin{aligned} &f(a + b + c) + f(b + c + d) + f(a - d) \\ &= f(a + b + d) + f(a + c + d) + f(b - c). \end{aligned}$$

Our aim is to determine the set $S(R, G)$ of the solutions in the case of certain structures R and G .

2. General investigations

In any commutative ring $R(+, \cdot)$ an elementary identity of RAMANUJAN ([5], p. 385) implies the following assertion. If $p_k(x) := x^k$ ($x \in R$, $k \in \mathbb{N}$) then p_2 and p_4 belong to $S(R, R)$, that is, $f = p_k$ ($k = 2, 4$) fulfils the functional equation (3). If

$$a_i : R \rightarrow G \quad (i = 1, 2)$$

are additive functions (i.e., $a_i(x + y) = a_i(x) + a_i(y)$ for any $(x, y \in R)$) and $a_0 \in G$ then the function

$$(4) \quad f(x) := a_2(x^4) + a_1(x^2) + a_0 \quad (x \in R)$$

$(f : R \rightarrow G)$ is an element of $S(R, G)$.

The converse is not generally true, for example in the case of $R = \mathbb{Z}$ and $G = \mathbb{R}$ there exist solutions that cannot be written in the form (4) (DARÓCZY-HAJDU [4]).

Theorem 1. *Let $R(+, \cdot)$ be a commutative ring with identity ($e := 1$). If f belongs to $S(R, G)$ then*

$$(5) \quad f(-x) = f(x) \quad \text{for any } x \in R$$

and

$$(6) \quad \begin{aligned} & f(tz) + f[tz(y+1) + t(y^2 + y + 1)] + f[tzy + t(y^2 + y + 1)] \\ & - f[tz(y+1) + t(y^2 + 2y)] - f[tz + t(2y + 1)] \\ & - f[tzy + t(y^2 - 1)] = 0 \end{aligned}$$

for any $t, z, y \in R$.

PROOF. By interchanging variables b and c in (3) we have $f(b - c) = f(c - b)$, which implies (5) for all $x \in R$ (with the notation $x := b - c$). For any $t, x, y \in R$ let

$$A := \begin{pmatrix} txy & tx \\ ty & t \end{pmatrix} \in \text{Mat}^*(2, R).$$

Then from (3)

$$\begin{aligned} & f(txy + tx + ty) + f(tx + ty + t) + f(txy - t) \\ & = f(txy + tx + t) + f(txy + ty + t) + f(tx - ty) \end{aligned}$$

follows, which implies, with the notation $z := x - y$ (i.e., $x = z + y$), the functional equation (6). $\square$

Theorem 2. *If R is a field and $f \in S(R, G)$ then (5) holds, and for any $x, y, t \in R$*

$$(7) \quad \begin{aligned} & f(x) + f[x(y+1) + t(y^2 + y + 1)] + f[xy + t(y^2 + y + 1)] \\ & - f[x(y+1) + t(y^2 + 2y)] - f[x + t(2y + 1)] \\ & - f[xy + t(y^2 - 1)] = 0. \end{aligned}$$

PROOF. If $t \neq 0$ then let $z := t^{-1}x$ in (6), where $x \in R$ is arbitrary. This implies the validity of (7) for any $x, y \in R$ and $t \neq 0$ ($t \in R$). It is easy to verify that (7) also holds for $t = 0$, and this completes the proof of the theorem. $\square$

The above theorem shows that another type of functional equations can be used to solve our problem. Functional equation (7) can also be written in the following form: let $f_1 = f_2 = f$, $f_3 = f_4 = f_5 = -f$ and for any $y \in R$ fixed

$$(8.0) \quad \begin{aligned} \varphi_{1,y}(x) &= \varphi_{3,y}(x) := (y+1)x \\ \varphi_{2,y}(x) &= \varphi_{5,y}(x) := yx \\ \varphi_{4,y}(x) &:= x \quad (x \in R) \end{aligned}$$

and

$$(8.1) \quad \begin{aligned} \psi_{1,y}(t) &= \psi_{2,y}(t) := (y^2 + y + 1)t \\ \psi_{3,y}(t) &:= (y^2 + 2y)t \\ \psi_{4,y}(t) &:= (2y + 1)t \\ \psi_{5,y}(t) &:= (y^2 - 1)t \quad (t \in R). \end{aligned}$$

With the above notation (7) implies

$$(9) \quad f(x) + \sum_{i=1}^5 f_i[\varphi_{i,y}(x) + \psi_{i,y}(t)] = 0$$

for any $x, t \in R$ and $y \in R$, where $f, f_i : R \rightarrow G$ ($i = 1, 2, 3, 4, 5$) are unknown functions and for a fix $y \in R$ the functions

$$\varphi_{i,y}, \psi_{i,y} : R \rightarrow R \quad (i = 1, 2, 3, 4, 5)$$

are additive, (that is, fulfil the Cauchy functional equation $a(x+y) = a(x) + a(y)$ ($x, y \in R$), $a : R \rightarrow R$). The type of functional equations of form (9) is known for a fix $y \in R$, this is the so-called “linear” functional equation (SZÉKELYHIDI [6], [7]), which can be solved generally under certain conditions. The following results deal with this problem.

Let S and G be Abelian groups, $n \in \mathbb{N}$, let φ_i, ψ_i be additive functions from S into S , and let

$$(10) \quad \text{rg}(\varphi_i) \subseteq \text{rg}(\psi_i) \quad (i = 1, 2, \dots, n+1).$$

If $f, f_i : G \rightarrow G$ ($i = 1, 2, \dots, n+1$) satisfy the “linear” functional equation

$$f(x) + \sum_{i=1}^{n+1} f_i[\varphi_i(x) + \psi_i(t)] = 0$$

for any $x, t \in G$ then $f : S \rightarrow G$ fulfils the Fréchet equation

$$(11) \quad \Delta_{y_1, y_2, \dots, y_{n+1}} f(x) = 0$$

for all $x, y_1, y_2, \dots, y_{n+1} \in S$, where Δ_y is the difference operator, i.e., for $f : S \rightarrow G$ and $y \in S$ $\Delta_y f(x) := f(x + y) - f(x)$ for any $x \in S$.

If G is a torsion-free Abelian group in which multiplication by any positive integer is bijective then G is a linear space over the field $\mathbb{Q}$ of the rationals. In this case the following theorem is true. If S and G are Abelian groups and G is a linear space over $\mathbb{Q}$ then $f : S \rightarrow G$ is the solution of the Fréchet equation (11) if and only if f is a polynomial of degree at most n .

The notion of polynomials on a group was introduced by S. MAZUR and W. ORLICZ [9], M. FRÉCHET [8], and G. VAN DER LIJN [10] and means the following. Let $k \in \mathbb{N}$, and let $A_k : G^k \rightarrow S$ be a k -additive (i.e., additive in all variables), and symmetric function. For $k = 0$ let $A_0 : G \rightarrow S$ be the constant function (i.e., there exists $a_0 \in S$ such that $A_0(x) = a_0$ for any $x \in S$). In this case let

$$(12) \quad A_k^*(x) := A_k(x, x, \dots, x) \quad (x \in G, k = 0, 1, 2, \dots)$$

be the so-called diagonal of A_k . A function $p : G \rightarrow S$ is called a polynomial of degree at most n if there exist k -additive, symmetric functions A_k ($k = 0, 1, \dots, n$) such that

$$(13) \quad p = \sum_{k=0}^n A_k^*.$$

According to the above results, in the following we assume that

- (i) G is an Abelian group which is a linear space over $\mathbb{Q}$, and
- (ii) R is a field of characteristic zero.

Theorem 3. *If R is a field of characteristic zero, G is a linear space over the field $\mathbb{Q}$ of the rationals, and $f \in S(R, G)$ then there exist k -additive, symmetric functions $A_k : R^k \rightarrow G$ ($k = 4, 2, 0$, $G^0 := G$) such that*

$$(14) \quad f(x) = A_4^*(x) + A_2^*(x) + A_0^*$$

holds for any $x \in R$.

PROOF. In this case f satisfies the functional equation (7) for all $x, t, y \in R$. Let $y = 2$ in (7). Then

$$f(x) + f(3x + 7t) + f(2x + 7t) - f(3x + 8t) - f(x + 5t) - f(2x + 3t) = 0$$

for any $x, t \in R$, and with the notations of (8.0) and (8.1) the functional equation

$$(15) \quad f(x) + \sum_{i=1}^5 f_i[\varphi_{i,2}(x) + \psi_{i,2}(t)] = 0$$

holds for all $x, t \in R$. Since in (15)

$$\text{rg}(\varphi_{i,2}) = \text{rg}(\psi_{i,2}) = R \quad (i = 1, 2, 3, 4, 5)$$

in R , which is a field of characteristic zero, therefore f is a polynomial of degree at most 4, that is,

$$(16) \quad f(x) = \sum_{k=0}^4 A_k^*(x) \quad (x \in R),$$

where $A_k : R^k \rightarrow G$ are k -additive, symmetric functions. On the other hand, from (16) we get

$$(17) \quad f(-x) = A_4^*(x) - A_3^*(x) + A_2^*(x) - A_1^*(x) + A_0^*$$

for all $x \in R$, which implies, as a consequence of equations (5), (16) and (17),

$$f(x) = \frac{f(x) + f(-x)}{2} = A_4^*(x) + A_2^*(x) + A_0^*$$

for any $x \in R$. □

Theorem 3 does not state that the final form of f is (14), only that f has the representation (14). This however does not imply that the functions of the form (14) satisfy the functional equation (3). Therefore in the following we shall examine under which (other) conditions a function of the form (14) belongs to $S(R, G)$.

3. On the polynomial solutions

In the following R is a commutative ring with identity, and G is linear space over the field $\mathbb{Q}$ of the rationals. Let $A_k : R^k \rightarrow G$ ($k = 4, 2, 0$) be k -additive and symmetric functions given and, according to (14), let

$$(18) \quad f = A_4^* + A_2^* + A_0^* \quad (f : R \rightarrow G).$$

We shall examine what conditions are necessary and sufficient for the function f defined in (18) to belong to $S(R, G)$.

Theorem 4. *If the function f defined in (18) is in $S(R, G)$ then $A_k^* \in S(R, G)$ if $k = 4, 2, 0$.*

PROOF. The assertion is trivial for A_0^* . So if $f \in S(R, G)$ then $g := (f - A_0^*) \in S(R, G)$. On the other hand, note that if $g \in S(R, G)$ then the function

$$g_2(x) := g(2x) \quad (x \in R)$$

is an element of $S(R, G)$, too. This implies that the function

$$\begin{aligned} \frac{1}{12} [g(2x) - 4g(x)] &= \frac{1}{12} [A_4^*(2x) + A_2^*(2x) - 4A_4^*(x) - 4A_2^*(x)] \\ &= \frac{1}{12} [16A_4^*(x) + 4A_2^*(x) - 4A_4^*(x) - 4A_2^*(x)] = A_4^*(x) \quad (x \in R) \end{aligned}$$

also belongs to $S(R, G)$, from which $A_2^* \in S(R, G)$ follows, as well. $\square$

Theorem 5. $A_2^* \in S(R, G)$ if and only if

$$(19) \quad A_2^*(x) = A_2(x^2, 1) \quad (x \in R).$$

PROOF. Let $x \in R$ be arbitrary, and let

$$A = \begin{pmatrix} x^2 & x \\ x & 1 \end{pmatrix}.$$

Then $D_{A_2^*}(A) = 0$ holds if and only if

$$A_2^*(x^2 + 2x) + A_2^*(2x + 1) + A_2^*(x^2 - 1) = 2A_2^*(x^2 + x + 1).$$

By the binomial theorem for multiadditive functions (see [6] for details), from this

$$\begin{aligned} & A_2^*(x^2) + 2A_2(x^2, 2x) + A_2^*(2x) + A_2^*(2x) + 2A_2(2x, 1) + A_2^*(1) \\ & \quad + A_2^*(x^2) + 2A_2(x^2, -1) + A_2^*(-1) \\ & = 2A_2^*(x^2) + 4A_2(x^2, x) + 4A_2(x^2, 1) + 2A_2^*(x) + 4A_2(x, 1) + 2A_2^*(1) \end{aligned}$$

follows, whence

$$A_2^*(x) = A_2(x^2, 1),$$

so (19) holds. On the other hand, the Ramanujan identity implies that the function $x \mapsto A_2(x^2, 1)$ ($x \in R$) belongs to $S(R, G)$. $\square$

Theorem 6. $A_4^* \in S(R, G)$ if and only if

$$(20) \quad A_4^*(x) = A_4(x^4, 1, 1, 1) \quad (x \in R).$$

PROOF. Let $x \in R$ be arbitrary, and let

$$A = \begin{pmatrix} x^2 & x \\ x & 1 \end{pmatrix}, \quad \text{and} \quad A' = \begin{pmatrix} x^2 & -x \\ -x & 1 \end{pmatrix}.$$

Then $D_{A_4^*}(A) = 0$ and $D_{A_4^*}(A') = 0$ hold if and only if

$$(21) \quad A_4^*(x^2 + 2x) + A_4^*(2x + 1) + A_4^*(x^2 - 1) = 2A_4^*(x^2 + x + 1),$$

and

$$(22) \quad A_4^*(x^2 - 2x) + A_4^*(-2x + 1) + A_4^*(x^2 - 1) = 2A_4^*(x^2 - x + 1).$$

Adding the two equations then using the binomial theorem, and the fact that

$$A_4^*(a + b) + A_4^*(a - b) = 2A_4(a, a, a, a) + 12A_4(a, a, b, b) + 2A_4(b, b, b, b),$$

we have

$$\begin{aligned} (23) \quad 5A_4(x, x, x, x) &= 2A_4(x^2, x^2, x^2, 1) + 2A_4(x^2, 1, 1, 1) \\ &\quad - 2A_4(x^2, x^2, x, x) + A_4(x^2, x^2, 1, 1) \\ &\quad - 2A_4(x, x, 1, 1) + 4A_4(x^2, x, x, 1). \end{aligned}$$

If in (23) we put $2x$ for x , then subtract the equation from (23) multiplied by 2^6 we obtain

$$(24) \quad 10A_4(x, x, x, x) = 5A_4(x^2, 1, 1, 1) - 5A_4(x, x, 1, 1) \\ + 2A_4(x^2, x^2, 1, 1) + 8A_4(x^2, x, x, 1).$$

Now replacing x by $2x$ in (24) again, then subtracting the equation from (24) multiplied by 2^2 , we get

$$(25) \quad 5A_4(x, x, x, x) = A_4(x^2, x^2, 1, 1) + 4A_4(x^2, x, x, 1),$$

and similarly we get that expressions of the same degree must be equal. The equalities involving odd degrees follow from (21) and (22) similarly.

$$(26.1) \quad A_4(x^2, 1, 1, 1) = A_4(x, x, 1, 1), \\ A_4(x^2, x, 1, 1) = A_4(x, x, x, 1),$$

$$(26.2) \quad A_4(x^2, x^2, x, 1) = A_4(x^2, x, x, x), \\ A_4(x^2, x^2, x^2, 1) = A_4(x^2, x^2, x, x).$$

Putting $x + 1$ instead of x in (26.2), we have

$$A_4(x^2 + 2x + 1, x^2 + 2x + 1, x + 1, 1) = A_4(x^2 + 2x + 1, x + 1, x + 1, x + 1).$$

Using the binomial theorem and the above equations, we obtain

$$2A_4(x, x, x, x) = A_4(x^2, x^2, 1, 1) + A_4(x^2, x, x, 1),$$

which together with (25) gives

$$A_4(x, x, x, x) = A_4(x^2, x^2, 1, 1).$$

Now, using equation (26.1), we have

$$A_4(x, x, x, x) = A_4(x^4, 1, 1, 1),$$

which was to be proved. Again, the Ramanujan identity implies that the function $x \mapsto A_4^*(x^4, 1, 1, 1)$ ($x \in R$) is in $S(R, G)$. $\square$

4. The main result, problems

On the basis of the previous results, the following main result can be stated.

Theorem 7. *Let R be a field of characteristic zero, and G a linear space over the field of the rationals. Then $f \in S(R, G)$ if and only if there exist additive functions $a_i : R \rightarrow G$ ($i = 1, 2$), and $a_0 \in G$ such that*

$$(27) \quad f(x) = a_2(x^4) + a_1(x^2) + a_0$$

for all $x \in R$.

PROOF. Theorems 3, 4, 5, and 6 imply that if $f \in S(R, G)$ then there exist k -additive and symmetric functions $A_k : R^k \rightarrow G$ ($k = 4, 2, 0$) for which

$$f(x) = A_4^*(x) + A_2^*(x) + A_0^* = A_4(x^4, 1, 1, 1) + A_2(x^2, 1) + A_0$$

for all $x \in R$. With the notations $a_2(x) := A_4(x, 1, 1, 1)$, $a_1(x) := A_2(x, 1)$ and $a_0 := A_0$ ($x \in R$) we have (23). On the other hand, earlier we proved that the functions of the form (23) are elements of $S(R, G)$ indeed. $\square$

It is clear from the above investigations that results similar to Theorem 7 cannot be expected for an arbitrary commutative ring with identity. The case of fields of non-zero characteristic also seems to be worth examining. Our method breaks down for such fields, since the condition $\text{rg}(\varphi) \subseteq \text{rg}(\psi)$ does not hold for the additive functions in question. So we think the following problem is still open and interesting. Let G be a linear space over the field of the rationals. Is there any field R of non-zero characteristic such that the solution set $S(R, G)$ contains elements that are not polynomials?

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