

Projective modules over twisted group algebras of p -solvable groups

By ANDREI MARCUS (Cluj-Napoca)

Abstract. In this paper we generalize some results of P. FONG and I. M. ISAACS on modular representation theory of p -solvable groups to the case of twisted group algebras and the further to the case of strongly group graded algebras. For this, we use a categorical approach to Clifford theory.

1. Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$, and let G be a finite group. If V is an $\mathcal{O}$ -module, then either V is $\mathcal{O}$ -free of finite rank or $J(\mathcal{O})V = 0$ and V has finite dimension over k .

Let further $R = \bigoplus_{g \in G} R_g$ be a strongly G -graded $\mathcal{O}$ -algebra (that is, R_g is an $\mathcal{O}$ -summand of R , and $R_g R_h = R_{gh}$ for all $g, h \in G$). For a subset X of G we denote $R_X = \bigoplus_{x \in X} R_x$. If V is a (left) R -module, then $(R|V)\text{-mod}$ is the full subcategory of $R\text{-mod}$ consisting of direct summands of finite direct sums of copies of V . We say that V is *isotypic* if all the indecomposable direct summands of V are isomorphic.

The main result of this paper states as follows. Assume that G is p -solvable and let $1 = N_0 < N_1 < \dots < N_r = G$ be a chain of normal subgroups of G such N_i/N_{i-1} is either a p -group or a p' -group, for $i = 1, \dots, r$. Let also H be a Hall p' -subgroup of G . (This means that the index of H in G is a power of p ; it is well-known that such an H exists and that any p' -subgroup of G is contained in a G -conjugate of H .) Let finally M be an indecomposable R_1 -module.

Mathematics Subject Classification: 20C20, 20C25, 20C05, 16W50, 16D90.

Key words and phrases: p -solvable groups, twisted group algebras, strongly graded algebras, Clifford theory.

1.1. Theorem. (a) Let $V \in (R|R \otimes_{R_1} M)\text{-mod}$ be an indecomposable R -module. Then there is an indecomposable R_H -module $W \in (R_H|R_H \otimes_{R_1} M)\text{-mod}$ such that $V \simeq R \otimes_{R_H} W$.

(b) Let $W, W' \in (R_H|R_H \otimes_{R_1} M)\text{-mod}$ be indecomposable R_H -modules such that the restrictions $\text{Res}_{R_H \cap N_i}^{R_H} W, \text{Res}_{R_H \cap N_i}^{R_H} W'$ are isotypic for $i = 0, \dots, r-1$, and $R \otimes_{R_H} W \simeq R \otimes_{R_H} W'$. Then $R \otimes_{R_H} W$ is an indecomposable R -module, and there is $g \in N_G(H)$ such that $W' \simeq R_{gH} \otimes_{R_H} W$ (as R_H -modules).

These statements are graded versions of the main results of HUBERT FOTTNER and BURKHARDT KÜLSHAMMER [2]. Their proof rely on the theory of G -algebras and on some technical results on lifting idempotents with group actions involved. We shall give here a short proof using the categorical approach to the Clifford theory of indecomposable modules of [5] and [6], and induction, of course. The main results of the above papers will be used in Section 2 to show that it is enough to deal with projective modules over twisted group algebras, and the same results are needed in Section 3 to prove the theorem in this case by induction. It should be noted that in the case of ordinary group algebras 1.1.(a) is due to P. FONG [1], while (b) to I. M. ISAACS [3].

We refer the reader to [7] for general facts on graded rings and to [4] for results on twisted group algebras and projective representations. Our notations tend to follow those of [5] and [6], and we shall recall the needed facts in the next section.

2. Clifford theory for strongly graded algebras

Let $\mathcal{O}, k, G, R$ and M be as in the introduction. The purpose of this section is to show that in Theorem 1.1, R can be replaced by a twisted group algebra of G and k .

2.1. If $\alpha \in Z^2(G, k^*)$ then denote, as in [4], by $k^\alpha G$ the twisted group algebra having k -basis $\{\bar{g} \mid g \in G\}$ and multiplication $\bar{g}\bar{h} = \alpha(g, h)\overline{gh}$. Then $k^\alpha G$ is a G -graded algebra in an obvious way. If $\beta \in Z^2(G, k^*)$, then $k^\alpha G \simeq k^\beta G$ as G -graded k -algebras if and only if α and β are cohomologous.

If G is a p -group, then it is well-known that $Z^2(G, k^*) = 1$ and $k^\alpha G \simeq kG$ is a local ring with $k^\alpha G/J(k^\alpha G) \simeq k$.

If H is a subgroup of G , we shall denote $k^\alpha H = k^{\text{res}_H^G \alpha} H$, where $\text{res}_H^G \alpha \in H^2(H, k^*)$ is the restriction of α to H . If W is a $k^\alpha H$ -module and V is a $k^\alpha G$ -module, then $\text{Ind}_{k^\alpha H}^{k^\alpha G} W = k^\alpha G \otimes_{k^\alpha H} W$ is the induced module, and $\text{Res}_{k^\alpha H}^{k^\alpha G} V$ is the $k^\alpha G$ -module obtained by scalar restriction via the inclusion $k^\alpha H \hookrightarrow k^\alpha G$.

If N is a normal subgroup of G , it will be useful to regard $k^\alpha G$ as a G/N -graded algebra, where for $x = gN \in G/N$, $(k^\alpha G)_x = \bar{g}k^\alpha N$.

2.2. The *graded Jacobson radical* of R is, by definition, the intersection of maximal graded left ideals of R , and it coincides with the two-sided graded ideal $J_{\text{gr}}(R) = J(R_1)R = RJ(R_1)$. Then $R/J_{\text{gr}}(R)$ is a strongly graded k -algebra with $(R/J_{\text{gr}}(R))_1 \simeq R_1/J(R_1)$. It is well-known that $J_{\text{gr}}(R) \subseteq J(R)$, and if G is a p' -group, then $J_{\text{gr}}(R) = J(R)$.

2.3. *Remark.* The following observation will be crucial in the next section. Let N be a normal p -subgroup of G and H a p' -subgroup of G such that $G = HN$. Let $\alpha \in Z^2(G, k^*)$ and regard $k^\alpha G$ as a G/N -graded k -algebra. Let f denote the composition $k^\alpha H \hookrightarrow k^\alpha G \rightarrow k^\alpha G/J_{\text{gr}}(k^\alpha G)$ (where, using the G/N -grading, $J_{\text{gr}}(k^\alpha G) = J(k^\alpha N)J(k^\alpha G)$, and by the last statement of (2.2) we have $J_{\text{gr}}(k^\alpha G) = J(k^\alpha G)$ since G/N is a p' -group). Clearly, f is a homomorphism of G/N -graded k -algebras, and since $k^\alpha N$ is a local ring, we have that $k^\alpha G/J_{\text{gr}}(k^\alpha G)$ is a twisted group algebra, so there is $\beta \in Z^2(H, k^*)$ such that $k^\alpha G/J_{\text{gr}}(k^\alpha G) \simeq k^\beta H$. It follows immediately that f is an isomorphism of H -graded k -algebras. Consequently, we have the injective map $k^\alpha H \rightarrow k^\alpha G$ and the surjective map $k^\alpha G \rightarrow k^\alpha H$ given by the composition $k^\alpha G \rightarrow k^\beta H \xrightarrow{f} k^\alpha H$.

2.4. Consider now the indecomposable R_1 -module M and let

$$G_M = I_G(M) = \{g \in G \mid R_g \otimes_{R_1} M \simeq M \text{ in } R_1\text{-mod}\}$$

be the stabilizer (inertia group) of M . Let further $E = \text{End}_{R_1}(R \otimes_{R_1} M)^{\text{op}}$. Then it is well-known that E is a G -graded $\mathcal{O}$ -algebra (not necessarily strongly graded) with

$$E_g \simeq \{f \in E \mid f(R_x \otimes_{R_1} M) \subseteq R_{xg} \otimes_{R_1} M \text{ for all } x \in G\}.$$

In particular, $E_1 \simeq \text{End}_{R_1}(M)^{\text{op}}$, and moreover, for any subgroup H of G , $E_H \simeq \text{End}_{R_H}(R_H \otimes_{R_1} M)^{\text{op}}$.

2.5. Let $D = E/J_{\text{gr}}(E)$. Then, since E_1 is a local ring and k is algebraically closed, we have that D is a twisted group algebra of G_M and k , so there is $\alpha \in H^2(G_M, k^*)$ such that $D = k^\alpha G_M$ (see [5], Section 3 for details).

2.6. By [5, Theorem 4.1], we have that the additive functor

$$D \otimes_E \text{Hom}_R(R \otimes_{R_1} M, -): (R|R \otimes_{R_1} M)\text{-mod} \rightarrow (D|D)\text{-mod}$$

induces an isomorphism between the Grothendieck groups associated to these categories, where $(D|D)\text{-mod}$ is the category of (finitely generated) projective D -modules. Moreover, this functor commutes with induction from subgroups, restriction, truncation and conjugation (see [5, Theorem 4.1 and Remark 4.5.b]), and the functors

$$(-)_{G_M}: (R|R \otimes_{R_1} M)\text{-mod} \rightarrow (R_{G_M}|R_{G_M} \otimes_{R_1} M)\text{-mod}$$

and

$$R \otimes_{R_H} -: (R_{G_M}|R_{G_M} \otimes_{R_1} M)\text{-mod} \rightarrow (R|R \otimes_{R_1} M)\text{-mod}$$

induce isomorphisms between the Grothendieck groups of these categories ([5, Theorem 4.1.c]).

This implies (see [5], Corollary 4.4) that if $V \in (R|R \otimes_{R_1} M)\text{-mod}$ is indecomposable, then $\text{Res}_{R_1}^R V$ is an isotypic R_1 -module if and only if $G_M = G$.

The above discussion implies immediately that Theorem 1.1 is equivalent to the following theorem.

2.7. Theorem. *Assume that G is a p -solvable group, H is a Hall p' -subgroup of G , and $\alpha \in Z^2(G, k^*)$. Let $1 = N_0 < N_1 < \dots < N_r = G$ be a chain of normal subgroups of G such that N_i/N_{i-1} is either a p -group or a p' -group.*

(a) *Let V be a projective indecomposable $k^\alpha G$ -module. Then there is a simple $k^\alpha H$ -module W such that $V \simeq \text{Ind}_{k^\alpha H}^{k^\alpha G} W$.*

(b) *Let W, W' be simple $k^\alpha H$ -modules such that $\text{Res}_{k^\alpha(H \cap N_i)}^{k^\alpha H} W$ and $\text{Res}_{k^\alpha(H \cap N_i)}^{k^\alpha H} W'$ are isotypic for $i = 0, \dots, r-1$, and $\text{Ind}_{k^\alpha H}^{k^\alpha G} W \simeq \text{Ind}_{k^\alpha H}^{k^\alpha G} W'$ as $k^\alpha G$ -modules. Then $\text{Ind}_{k^\alpha H}^{k^\alpha G} W$ is an indecomposable (and*

projective) $k^\alpha G$ -module, and there is $g \in N_G(H)$ such that $k^\alpha H \bar{g} \otimes_{k^\alpha H} W$ as $k^\alpha H$ -modules.

The proof of this theorem will be given in the next section. It will require Clifford theory of projective modules, so we recall the basic result [6, Theorem 2.3] in the general context of this section.

2.8. Assume in addition that M is a projective R_1 -module, and let $S = M/J(R_1)M$ and $R' = R/J_{\text{gr}}(R)$. It follows that S is a simple R'_1 -module and $\text{End}_{R'}(R' \otimes_{R'_1} S)^{op} \simeq D$ as G_M -graded k -algebras. Moreover, there is a commutative diagram of categories

$$\begin{array}{ccc} (R|R \otimes_{R_1} M)\text{-mod} & \xrightarrow{\text{Hom}_R(R \otimes_{R_1} M, -)} & (E|E)\text{-mod} \\ R' \otimes_R - \downarrow & & \downarrow D \otimes_E - \\ (R'|R' \otimes_{R'_1} S) & \xrightarrow{\text{Hom}_{R'}(R' \otimes_{R'_1} S, -)} & (D|D)\text{-mod} \end{array}$$

and this diagram is also compatible with induction from subgroups.

3. A proof of Theorem 2.7

Assume that G is p -solvable, H a Hall p' -subgroup of G , and let $1 = N_0 < N_1 < \cdots < N_r = G$ be a chain of normal subgroups as in (2.7).

(a) We prove by induction on G that if $\alpha \in Z^2(G, k^*)$ and V is a projective $k^\alpha G$ -module, then there is a simple $k^\alpha H$ -module W such that $V \simeq \text{Ind}_{k^\alpha H}^{k^\alpha G} W$.

The statement is trivial if G is a p' -group (since then $H = G$) or if G is a p -group (then $H = 1$ and $V = k^\alpha G$ is the unique projective indecomposable $k^\alpha G$ -module).

Denote $N = N_1$ and assume first that $N \neq 1$ is a p' -group. Since N is a normal subgroup of G , we have that $N \subseteq H$. Since V is a projective indecomposable $k^\alpha G$ -module, by Clifford theory there is a projective (and simple) $k^\alpha N$ -module M such that $V \in (k^\alpha G | \text{Ind}_{k^\alpha N}^{k^\alpha G} M)\text{-mod}$. Let $I/N = (G/N)_M$ be the stabilizer of M . By (2.6) there is a projective $k^\alpha I$ -module V_0 such that $V \simeq \text{Ind}_{k^\alpha I}^{k^\alpha G} V_0$, and moreover $\text{Res}_{k^\alpha N}^{k^\alpha I} V_0$ is an isotypic $k^\alpha N$ -module. By (2.6) and (2.5) V_0 corresponds to a projective indecomposable $k^\beta(I/N)$ -module $\tilde{V}_0$, where $\beta \in Z^2(I/N, k^*)$. Denote

$H_0 = I \cap H$, so H_0/N is a Hall p' -subgroup of I/N . Since $|H_0/N| < |G|$, by induction there is a projective and simple $k^\beta H_0$ -module $\tilde{W}_0$ such that $\tilde{V}_0 \simeq \text{Ind}_{k^\beta(H_0/N)}^{k^\beta(I/N)} \tilde{W}_0$. Again by (2.6), $\tilde{W}_0$ corresponds to a projective $k^\alpha H_0$ -module $W_0 \in (k^\alpha H_0 | \text{Ind}_{k^\alpha N}^{k^\alpha H_0} M)\text{-mod}$ and $V_0 \simeq \text{Ind}_{k^\alpha H_0}^{k^\alpha I} W_0$. Denoting $W = \text{Ind}_{k^\alpha H_0}^{k^\alpha H} W_0$, we finally have

$$\begin{aligned} V &\simeq \text{Ind}_{k^\alpha I}^{k^\alpha G} V_0 \simeq \text{Ind}_{k^\alpha I}^{k^\alpha G} \text{Ind}_{k^\alpha H_0}^{k^\alpha I} W_0 \\ &\simeq \text{Ind}_{k^\alpha H}^{k^\alpha G} \text{Ind}_{k^\alpha H_0}^{k^\alpha H} W_0 \simeq \text{Ind}_{k^\alpha H}^{k^\alpha G} W. \end{aligned}$$

Assume now that $N \neq 1$ is a p -group. Since V is a projective indecomposable $k^\alpha G$ -module, there is a projective indecomposable $k^\alpha N$ -module M such that $V \in (k^\alpha G | \text{Ind}_{k^\alpha N}^{k^\alpha G} M)\text{-mod}$. But N is a p -group, so $M \simeq k^\alpha N$ is the unique projective indecomposable $k^\alpha N$ -module, hence M is G/N -invariant. By (2.6) and (2.7) V corresponds to a projective indecomposable $k^\beta(G/N)$ -module $\tilde{V}$, where $\beta \in Z^2(G/N, k^*)$. We have that $|G/N| < |G|$ and HN/N is a Hall p' -subgroup of G/N . By induction it follows that there is a projective simple $k^\beta(HN/N)$ -module $\tilde{W}$ such that $\tilde{V} \simeq \text{Ind}_{k^\beta(HN/N)}^{k^\beta(G/N)} \tilde{W}$. By (2.6) again, $\tilde{W}$ corresponds to a projective indecomposable $k^\alpha(HN)$ -module $P \in (k^\alpha(HN) | \text{Ind}_{k^\alpha N}^{k^\alpha(HN)} M)\text{-mod}$ such that $V \simeq \text{Ind}_{k^\alpha(HN)}^{k^\alpha G} P$.

Since $\tilde{W}$ is a simple $k^\beta(HN/N)$ -module, by (2.8) it corresponds to a simple module W over $k^\alpha(HN)/J_{\text{gr}}(k^\alpha(HN))$ -module, where $k^\alpha(HN)$ is regarded as a HN/N -graded algebra. By Remark 2.3, W is a projective simple $k^\alpha H$ -module. Clearly, $k^\alpha(HN) \otimes_{k^\alpha H} W$ is a projective $k^\alpha(HN)$ -module, and multiplication induces an epimorphism $\text{Ind}_{k^\alpha H}^{k^\alpha(HN)} W \rightarrow W$. By the commutativity of the diagram (2.8) we have that P is the projective cover of W (where W is regarded as a simple $k^\alpha(HN)$ -module), so P is a direct summand of $\text{Ind}_{k^\alpha H}^{k^\alpha(HN)} W$. Since $W \simeq P/J(k^\alpha H)P$, we have that $\dim_k P = |N|\dim_k W$; comparing dimensions, it follows that $P \simeq \text{Ind}_{k^\alpha H}^{k^\alpha N} W$, hence $V \simeq \text{Ind}_{k^\alpha H}^{k^\alpha G} W$.

(b) We use again induction on G , and we see that the statements are trivial if G is a p' -group or a p -group.

Suppose first that $N = N_1 \neq 1$ is a p' -group. By the assumption on isotopy, there is a G/N -invariant $k^\alpha N$ -module M such that W and W' belong to the same category $(k^\alpha H | \text{Ind}_{k^\alpha N}^{k^\alpha G} M)\text{-mod}$. Denoting

$V = \text{Ind}_{k^\alpha H}^{k^\alpha G} W \simeq \text{Ind}_{k^\alpha H}^{k^\alpha G} W'$, we have that V belongs to the category $(k^\alpha G | \text{Ind}_{k^\alpha N}^{k^\alpha G} M)\text{-mod}$. By (2.6) and (2.4) V corresponds to a projective $k^\beta(G/N)$ -module, W and W' correspond to projective simple $k^\beta(H/H)$ -modules $\tilde{W}$ and $\tilde{W}'$ respectively, such that

$$\tilde{V} \simeq \text{Ind}_{k^\beta(H/N)}^{k^\beta(G/N)} \tilde{W} \simeq \text{Ind}_{k^\beta(H/N)}^{k^\beta(G/N)} \tilde{W}',$$

and $\text{Res}_{k^\beta(H \cap N_i/N)}^{k^\beta(H/N)} \tilde{W}, \text{Res}_{k^\beta(H \cap N_i/N)}^{k^\beta(H/N)} \tilde{W}'$ are isotypic for $i = 1, \dots, r-1$.

Since $|G/N| < |G|$ and H/N is a Hall p' -subgroup of G/N , by the induction hypothesis we have that $\tilde{V}$ is an indecomposable $k^\beta(G/N)$ -module, and there is $gN \in N_{G/N}(H/N)$ such that

$$\tilde{W}' \simeq k^\beta(H/N) \overline{gN} \otimes_{k^\beta(H/N)} \tilde{W}.$$

By (2.6) it follows that V is an indecomposable $k^\alpha G$ -module and $W' \simeq k^\alpha H \bar{g} \otimes_{k^\alpha H} W$.

Finally, assume that $N \neq 1$ is a p -group. By Remark 2.3 we have an epimorphism $k^\alpha(HN) \rightarrow k^\alpha H$, and regard W, W' as simple $k^\alpha(HN)$ -modules by scalar restriction. Let $P = P(W)$ and $P' = P(W')$ be the projective covers of W and W' respectively, so P and P' are projective indecomposable $k^\alpha(HN)$ -modules. By the last part of the proof of (a) we have that $P \simeq \text{Ind}_{k^\alpha H}^{k^\alpha(HN)} W$ and $P' \simeq \text{Ind}_{k^\alpha H}^{k^\alpha(HN)} W'$. Moreover, by assumption

$$V \simeq \text{Ind}_{k^\alpha(HN)}^{k^\alpha G} P \simeq \text{Ind}_{k^\alpha(HN)}^{k^\alpha G} P'.$$

Clearly, V belongs to $(k^\alpha G | \text{Ind}_{k^\alpha N}^{k^\alpha G} M)\text{-mod}$ while P and P' belong to the category $(k^\alpha(HN) | \text{Ind}_{k^\alpha N}^{k^\alpha(HN)} M)\text{-mod}$, where $M \simeq k^\alpha N$ is the unique projective indecomposable $k^\alpha N$ -module. (Hence M is G -invariant and we do not have to use the assumption on isotypy when we restrict to normal p -groups.) By (2.6) and (2.4) V corresponds to a projective $k^\beta(G/N)$ -module $\tilde{V}$, while W and W' correspond to projective $k^\beta(HN/N)$ -modules $\tilde{W}$ and $\tilde{W}'$ respectively, where $\beta \in Z^2(G/N, k^*)$. Moreover, by assumption it follows that

$$\tilde{V} \simeq \text{Ind}_{k^\beta(HN/N)}^{k^\beta(G/N)} \tilde{W} \simeq \text{Ind}_{k^\beta(HN/N)}^{k^\beta(G/N)} \tilde{W}'.$$

Since $|G/N| < G$ and HN/N is a Hall p' -subgroup of G/N , the induction hypothesis can be applied, hence $\tilde{V}$ is an indecomposable $k^\beta(G/N)$ -module, and there is $gN \in N_{G/N}(HN/N)$ (so $g \in N_G(H)$) such that

$\tilde{W}' \simeq k^\beta(H/N)\overline{gN} \otimes_{k^\beta(H/N)} \tilde{W}$. By (2.6) it follows that V is an indecomposable $k^\alpha G$ -module and $P' \simeq k^\alpha(HN)\bar{g} \otimes_{k^\alpha(HN)} P$ as $k^\alpha(HN)$ -modules. Consequently, by Remark 2.3,

$$W' \simeq P'/J(k^\alpha N)P' \simeq k^\alpha H\bar{g} \otimes_{k^\alpha H} P/J(k^\alpha H)P \simeq k^\alpha H\bar{g} \otimes_{k^\alpha H} W$$

as $k^\alpha H$ -modules.

Acknowledgements. The author thanks the referee for his observations which improved the presentation of the paper.

References

- [1] P. FONG, Solvable groups and modular representation theory, *Trans. Amer. Math. Soc.* **103** (1962), 484–494.
- [2] H. FOTTNER and B. KÜLSHAMMER, Remarks on G -algebras, *Preprint*, 1997.
- [3] I. M. ISAACS, Fong Characters in π -separable groups, *J. Algebra* **99** (1986), 83–107.
- [4] G. KARPILOVSKI, Projective representations of finite groups, *Marcel Dekker, New York*, 1985.
- [5] A. MARCUS, Static modules and Clifford theory for strongly graded rings, *Publicaciones Math.* **42** (1993), 303–314.
- [6] A. MARCUS, Clifford theory for projective modules over strongly graded rings, *Comm. Algebra* **23** (1995), 4393–4404.
- [7] C. NĂSTĂSESCU and F. VAN OYSTAEYEN, Graded ring theory, *North Holland, Amsterdam*, 1982.

ANDREI MARCUS
 “BABEŞ-BOLYAI” UNIVERSITY
 FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
 STR. M. KOGALNICEANU NR. 1
 3400 CLUJ-NAPOCA
 ROMANIA

E-mail: marcus@math.ubbcluj.ro

(Received August 13, 1997; revised January 19, 1998)