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On the power values of polynomials

By A. BÉRCZES (Debrecen), B. BRINDZA (Debrecen)
and L. HAJDU (Debrecen)

Abstract. In this paper we give a new, generalized version of a result of Brindza, Evertse and Győry, concerning superelliptic equations.

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n and b be a nonzero integer. For effective upper bounds obtained by Baker's method for the exponent z in the equation

$$f(x) = by^z, \quad x, y, z \in \mathbb{Z} \text{ with } |y| > 1,\ z > 1 \tag{1}$$

we refer to [T], [ST], [Tu1], [Tu2], [ShT], [B1], [BEGy], [Bu].

For a polynomial P let $M(P)$ denote the Mahler height of it (cf. [M]). The purpose of this paper, which is related to a recent observation of Brindza on the number of the solutions of a generalization of the Ramanujan–Nagell equation [B3], is to derive a bound for z which is polynomial in $M(f)$. For brevity write $M = M(f)$.

Theorem. *If f has at least two distinct zeros, then*

$$z < cM^{3n} \log^3 |2b|,$$

where c is an effectively computable constant depending only on n.

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Remarks. If f is an irreducible monic and $b = 1$ then this inequality was proved by BRINDZA, GYŐRY and EVERTSE with different constants (see [BEGy], Th. 4). Moreover, if $n > 2$ and f is irreducible then a profound result of GYŐRY (cf. [Gy1] or [Gy2]) makes it possible to substitute cM^{3n} by an effective constant depending only on the discriminant of f.

1. Auxiliary results

To prove our Theorem, we need two lemmas. In what follows, for any non-zero algebraic number α, $h(\alpha)$ and $H(\alpha)$ denotes the logarithmic height and the classical (ordinary) height of α, respectively.

Lemma 1. *Let $\mathbb{K}$ be an algebraic number field of degree n and denote by R and r the regulator and the unit rank of $\mathbb{K}$, respectively. There exists a fundamental sytem of units $\varepsilon_1, \dots, \varepsilon_r$ for $\mathbb{K}$ so that*

$$h(\varepsilon_i) \leq c^* R, \quad i = 1, \dots, r$$

where c^* is an effectively computable constant depending only on n.

PROOF. This statement is a consequence of Lemma 1 in [BGy]. For other versions of this result cf. [B2] or [H]. □

Lemma 2. *Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers and let $A_1, \dots, A_n$ be positive real numbers with $A_i \geq \max\{H(\alpha_i), e\}$ for $i = 1, \dots, n$. Furthermore, let $b_1, \dots, b_n$ be rational integers with $\alpha_1^{b_1} \dots \alpha_n^{b_n} \neq 1$ and suppose that B is a positive real number satisfying $B \geq \max_{i=1,\dots,n} |b_i|$ and $B \geq e$. Now we have*

$$|\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1| \geq B^{-c' \log A_1 \dots \log A_n},$$

where c' is an effectively computable constant depending only on n and on the degree of $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ over $\mathbb{Q}$.

PROOF. This is Theorem 1.2 in [PW]. □

2. Proof of the Theorem

We have two cases to distinguish.

First we assume that f has an irreducible factor $P \in \mathbb{Z}[x]$ of degree $t \geq 2$. Let α be a zero of P, moreover, let R, h, D and r be the regulator, class number, discriminant and unit rank of the field $\mathbb{K} = \mathbb{Q}(\alpha)$, respectively. In the sequel, $c_1, c_2, \dots$ will denote effectively computable positive constants depending only on n. The well-known inequalities

$$hR \leq \sqrt{|D|}\,(\log|D|)^{n-1}, \quad \text{(cf. e.g. [L])}$$

and

$$|D| \leq n^n M(P)^{2n-2} \leq n^n M^{2n-2} \quad \text{(cf. [M])}$$

imply

$$hR < c_1 M^n. \tag{2}$$

Let a denote the leading coefficient of f and $\beta_1, \dots, \beta_n$ be the zeros of $g(x) = a^{n-1} f(\frac{x}{a})$. Set

$$\Delta(g) = \prod_{\beta_i \neq \beta_j} (\beta_i - \beta_j)^2,$$

and write g in the form $g(x) = P_1^{k_1}(x) P_2(x)$ where P_1 and P_2 are relatively prime polynomials in $\mathbb{Z}[x]$ and P_1 is an irreducible monic of degree t; (actually $P_1(x) = a^t P(\frac{x}{a})$). Let $\beta_1, \dots, \beta_t$ be the zeros of P_1 and (x, y) be an arbitrary, however, fixed solution to (1). The g.c.d. of the principal ideals $\langle ax - \beta_1 \rangle$ and $\langle g(ax)(ax - \beta_1)^{-k_1} \rangle$ divides $\Delta^n(g)$, therefore, there are integral ideals A, B, C in $\mathbb{K}$ so that

$$A\langle ax - \beta_1 \rangle = BC^w \quad \text{where } w = \frac{z}{(z, k_1)}, \tag{3}$$

furthermore,

$$\max\{N_{\mathbb{K}/\mathbb{Q}}(A), N_{\mathbb{K}/\mathbb{Q}}(B)\} \leq |a \cdot b \cdot \Delta(g)|^{n^2}.$$

Hence, by a well-known inequality (cf. for example [Gy3], Lemma 3) and by (2), the ideals A^h and B^h have generators α and β, respectively, with

$$\max\left\{\overline{|\alpha|}, \overline{|\beta|}\right\} \leq \exp(c_2 M^{n-1} (\log M)^n \log|2b|).$$

The relation (3) can be written as

$$\alpha(ax-\beta_1)^h = \varepsilon\beta\gamma^w$$

where γ is a generator of C^h and ε is a unit. Let $\varepsilon_1,\dots,\varepsilon_r$ be a fundamental system of units for $\mathbb{K}$ satisfying Lemma 1. Then we can express ε as $\varepsilon = \rho\varepsilon_1^{l_1}\dots\varepsilon_r^{l_r}$ where ρ is a root of unity and we may assume that $\max\limits_{1\le i\le r}|l_i| < w$ (the remaining factors, if any, are incorporated in γ).

If $|ax| \le M(g)+1$ then

$$2^z \le |y|^z \le (2M(g)+1)^n$$

and the Theorem is proved. Otherwise, $|ax| > M(g)+1$ and $|ax-\beta_i| > 1$, $i=1,\dots,n$ implies

$$|ax-\beta_i| \le |a^{n-1}by^z|, \quad i=1,\dots,n,$$

$$|a^{n-1}by^z|^h \ge \max_{1\le i\le t}|ax-\beta_i|^h \ge \overline{|\varepsilon_1|}^{-nw}\dots\overline{|\varepsilon_r|}^{-nw}\overline{|\alpha|}^{-n}\overline{|\beta|}^{-n}\overline{|\gamma|}^{w}$$

and

$$\overline{|\gamma|} \le |a^{n-1}b|^{\frac{h}{w}}|y|^{nh}\overline{|\alpha|}^{\frac{n}{w}}\overline{|\beta|}^{\frac{n}{w}}\prod_{i=1}^{r}\overline{|\varepsilon_i|}^{n}.$$

If $w < nh$ then by $0.056 < R$ (cf. [Z]) we obtain $w < 20nhR$ and

$$z < c_3M^{n-1}(\log(2M))^{n-1}.$$

In case of $w \ge nh$

$$\overline{|\gamma|} \le M|b|^{\frac{1}{n}}|y|^{nh}\overline{|\alpha|}\,\overline{|\beta|}\prod_{i=1}^{r}\overline{|\varepsilon_i|}^{n},$$

and we get

$$\log H\left(\frac{\gamma}{\gamma^{(2)}}\right) \le c_4\log|2b|M^{n-1}(\log(2M))^n\log|y|.$$

We may assume that $|ax| \ge \frac{1}{2}|y|^{\frac{z}{n}}$. Indeed, otherwise $\max\limits_{1\le i\le n}|ax-\beta_i| \ge |y|^{\frac{z}{n}}$ yields

$$|ax| \ge |y|^{\frac{z}{n}} - M(g)$$

and the Theorem is proved. Supposing

$$\frac{|\beta_i-\beta_j|}{|ax-\beta_i|}\geq\frac{|\beta_2-\beta_1|}{|ax-\beta_2|},\quad 1\leq i,j\leq t,\ i\neq j$$

we have

$$\prod_{\substack{1\leq i,j\leq t\\ \beta_i\neq\beta_j}}\frac{|\beta_i-\beta_j|}{|ax-\beta_i|}\leq\frac{|\Delta(g)|\cdot 2^n}{|y|^z}.$$

Then

$$\frac{|\beta_2-\beta_1|}{|ax-\beta_2|}\leq|y|^{-\frac{z}{4}},$$

or else we can derive a bound for z better than stated in the Theorem. Avoiding the trivial case $\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h=1$, whenever $\frac{1}{2}|y|^{\frac{z}{n}}\leq|\Delta(g)|^{n^2}$ we obtain

$$\log\left|\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h-1\right|\leq\log\left(h\left|\frac{ax-\beta_1}{ax-\beta_2}-1\right|\right)\leq-\frac{z}{8}\log|y|.$$

Finally, Lemma 2 yields

$$\begin{aligned}0\neq\left|\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h-1\right|&=\left|\left(\frac{\varepsilon_1}{\varepsilon_1^{(2)}}\right)^{l_1h}\cdots\left(\frac{\varepsilon_r}{\varepsilon_r^{(2)}}\right)^{l_rh}\frac{\beta/\alpha}{\beta^{(2)}/\alpha^{(2)}}\left(\frac{\gamma}{\gamma^{(2)}}\right)^{wh}-1\right|\\&\geq\exp\left(-c_5\log|2b|M^{3n-3}(\log|2M|)^{3n-1}\log|y|\log w\right)\}\end{aligned}$$

and the comparision of the upper and lower bounds completes the proof (in the first case).

In the easier second case all the zeros of g are integral. Let k_i denote the multiplicities of β_i, $i=1,2$.

Repeating the argument one can have

$$u_i(ax-\beta_i)=v_iy_i^w$$

where $w=\frac{z}{(a,k_1k_2)}$ and $u_i,v_i,y_i\in\mathbb{Z}$, $|y_i|>1$, $i=1,2$.

To derive a bound for w from the equation

$$Ay_1^w - By_2^w = C$$

$(A = u_2v_1,\ B = u_1v_2,\ C = u_1u_2(\beta_2 - \beta_1))$ one can apply Lemma 2 again, and we have

$$\frac{z}{\log z} \le c_6 \log M \log |2b|,$$

and the Theorem is proved. □

References

[B1] B. Brindza, On S-integral solutions of the equation $y^m = f(x)$, *Acta Math. Hung.* **44** (1984), 133–139.

[B2] B. Brindza, On the generators of S-unit groups in algebraic number fields, *Bull. Austral Math. Soc.* **43** (1991), 325–329.

[B3] B. Brindza, On the generalized Ramanujan-Nagell equation, *Publ. Math. Debrecen* (*to appear*).

[BEGy] B. Brindza, J.-H. Evertse and K. Győry, Bounds for the solutions of some diophantine equations in terms of the discriminants, *J. Austral Math. Soc.* **51** (1991), 8–26.

[Bu] Y. Bugeaud, Sur la distance entre deux puissances pures, *C. R. Acad. Sci. Paris* **322**, Série I (1996), 1119–1121.

[BGy] Y. Bugeaud and K. Győry, Bounds for the solutions of unit equations, *Acta Arith.* **74** (1996), 67–80.

[Gy1] K. Győry, Sur les polynômes à coefficients entiérs et de discriminant donné, *Acta Arith.* **23** (1973), 419–426.

[Gy2] K. Győry, Sur les polynômes à coefficients entiérs et de discriminant donné II., *Publ. Math. Debrecen* **21** (1974), 125–144.

[Gy3] K. Győry, On the solutions of linear diophantine equations in algebraic integers of bounded norm, *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* **22/23** (1980), 225–233.

[H] L. Hajdu, A quantitative version of Dirichlet's S-unit theorem in algebraic number fields, *Publ. Math. Debrecen* **42** (1993), 239–246.

[L] H. W. Lenstra Jr., Algorithms in algebraic number theory, *Bull. Amer. Math. Soc.* **26** (1992), 211–244.

[M] K. Mahler, An inequality for the discriminant of a polynomial, *Michigan Math. J.* **11** (1964), 257–262.

[PW] P. Philippon and M. Waldschmidt, Lower bounds for linear forms in logarithms, New Advances in Transcendence Theory (A. Baker, ed.), *Cambridge Univ. Press*, *Cambridge*, 1988, 280–312.

[ST] A. Schinzel and R. Tijdeman, On the equation $y^m = P(x)$, *Acta Arith.* **31** (1976), 199–204.

[ShT] T.N. Shorey and R. Tijdeman, Exponential diophantine equations, *Cambridge Univ. Press*, *Cambridge*, 1986.

[T] R. Tijdeman, Applications of the Gelfond-Baker method to rational number theory, Topics in Number Theory, Proceedings of the Conference held in Debrecen 1974, Colloq. Math. Soc. János Bólyai 13, *North-Holland, Amsterdam*, 399–416.

[Tu1] J. Turk, Polynomial values and almost powers, *Michigan Math. J.* **29** (1982), 213–220.

[Tu2] J. Turk, On the difference between perfect powers, *Acta Arith.* **45** (1986), 289–307.

[Z] R. Zimmert, Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung, *Invent. Math.* **62** (1981), 367–380.

A. BÉRCZES, B. BRINDZA, L. HAJDU
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

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