

On automorphism groups of simple arguesian lattices

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Abstract. Let $\mathfrak{G}$ be a group. In this paper we prove that there exists a *simple arguesian* lattice M whose automorphism group is isomorphic to $\mathfrak{G}$.

A lattice L is called *interval finite*, if every interval of L is finite. In this note we give a new proof of a theorem of CHRISTIAN HERRMANN [3]. This theorem was proved by G. GRÄTZER and E. T. SCHMIDT [2] for finite groups and later by CHRISTIAN HERRMANN [3] in the present form.

Theorem. *Every group $\mathfrak{G}$ can be represented as the automorphism group of an interval finite, simple, arguesian lattice M .*

Let $\mathfrak{G}$ be a given group. By R. FRUCHT [1], there exists an undirected graph $\langle V, E \rangle$ with no loops whose automorphism group is isomorphic to $\mathfrak{G}$ (that is, V is a set and the set E of edges is a subset of two-elements subsets of V). We begin our construction with this graph.

We consider first a vector space $\mathfrak{V}$ over the two element field Z_2 with a basis V' . We assume that V and V' have the same cardinality, i.e. $|V| = |V'|$. Then we can identify the vertices of the graph with the basis elements of this vector space, that means, we can consider the elements $v_0, v_1, v_2, \dots$ of V as the basis elements of the vector space $\mathfrak{V}$. Let A be the lattice of all finitely generated subspaces of the vector space $\mathfrak{V}$. This lattice A is obviously a simple, atomistic, arguesian lattice. The vector

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space $\mathfrak{V}$ is over the two element field Z_2 , consequently every line contains three points. The subspace generated by v_i will be denoted by the same letter v_i . The lattice A has the following three types of atoms:

1. The atoms $v_i, i \in I$ (i.e. the elements of the basis), these form the set V and I an arbitrary index set;
2. Consider the third point $v_i + v_j$ ($i, j \in I$) of the line $\overline{v_i, v_j}$ spanned by v_i and v_j . Some of these $v_i + v_j$ -s correspond to edges of the graph (i.e. $\{v_i, v_j\}$ is an edge), in this case $v_i + v_j$ will be denoted by v_{ij} . All these atoms form a subset W ;
3. All other atoms.

We consider the given $\mathfrak{G}$ as a subgroup of the automorphism group of A . To the vertices of the Frucht graph correspond the atoms $v_i \in V, i \in I$ and to the edges $\{v_i, v_j\}$ correspond the atoms v_{ij} , these determine the edges in V . Obviously, every permutation of the v_i -s can be extended to an automorphism of A and every automorphism of A is determined by its restriction to the basis V . Indeed, if α and β are two automorphisms of A such that their restrictions to V are the same, then the restriction of $\gamma = \alpha\beta^{-1}$ is the identity map ϵ of V . By any extension of ϵ (i.e. automorphism with the property that its restriction to V is ϵ) the atoms v_i and v_j are fixed elements, consequently $v_i + v_j$ must be fixed. Similarly, $(v_i + v_j) + v_k$ must be a fixed element. In this way we get that by an extension of ϵ all atoms are fixed elements which means that this extension is the identity mapping of A . It follows that all automorphisms with the property that V and W are invariant form a group isomorphic to $\mathfrak{G}$. To ensure that we have no more automorphisms than the graph we must *label* the vertices and the edges, i.e. the atoms $v_i \in V$ and $v_{ij} \in W$. This will be done by lattices which are glued to A . The idea of the gluing is the following. The ideal $(v_i]$ of A has two elements. We will define a special lattice F_1 with a two element dual ideal D_1 which is therefore isomorphic to $(v_i]$. Similarly, for every $v_{ij} \in W$ we use a lattice F_2 with the dual ideal D_2 . For every $i \in I$ we consider an isomorphic copy F_1^i of F_1 with the dual ideal D_1^i and similarly the lattices $F_2^{ij} \cong F_2$ with the dual ideal D_2^{ij} . We can apply the gluing construction for the lattices A, F_1^i and F_2^{ij} simultaneously, identifying the ideal $(v_i]$ with D_1^i and $(v_{ij}]$ with D_2^{ij} . On this way we get a join-semilattice and M is the arguesian lattice generated by this configuration. First we define the lattices F_1, F_2 . We give the

description of M as a sublattice of a vectorspace lattice and prove that this is a simple arguesian lattice with the given automorphism group.

N is the chain of all nonnegative integers and N^* denotes the chain of the nonpositive integers. Take the direct product $\mathfrak{C}_2 \times N^*$, (where $\mathfrak{C}_2$ denotes the two element lattice). In this direct product for every $i \in N$ the elements $(0, -i - 1)$, $(1, -i - 1)$, $(0, -i)$, $(1, -i)$ form a “covering square” (isomorphic to $\mathfrak{C}_2 \times \mathfrak{C}_2$). Into these “covering squares”, for $i = 0, 1, \dots$ we insert one more element z_i so that a copy of $\mathfrak{M}_3$, the five element non distributive modular lattice, is obtained. The resulting lattice is F_1 , see Figure 1a. The lattice F_2 is similar but we don’t insert z_0 , into the first “covering square”, see Figure 1b. The dual ideal consisting of $(0, 0)$ and $(1, 0)$ etc. of F_1 is D_1 . We use isomorphic copies of F_1 and F_2 to label the v_i -s and the v_{ij} -s.

Figure 1a

Figure 1b

F_1 is a simple arguesian lattice and it has exactly one nontrivial automorphism α , where $\alpha(z_0) = (0, 0)$ and $\alpha(0, 0) = (z_0)$. F_2 is a rigid (has no nontrivial automorphism) arguesian lattice, its congruence lattice is the four element Boolean lattice.

We define our lattice M as a sublattice of a vectorspace lattice $K = L(\mathfrak{W})$ of a vectorspace $\mathfrak{W}$ over Z_2 . Take the set $\{u_j^k, v_j; j \in I, k \in N\}$ as a basis of $\mathfrak{W}$. Let z_j^k be the third point of the line spanned by u_j^k and v_j . Define the following subspaces, (where $[X]$ denotes the subspace spanned

by the set X): $\mathbf{o} = [u_j^k; j \in I, k \in N]$, $\mathbf{v}_i = [v_i, u_j^k; j \in I, k \in N] = [v_i, \mathbf{o}]$. The convex sublattice of K , generated by (as lattice) $\mathbf{v}_i$ -s form a sublattice isomorphic to A , we identify A with this sublattice.

Set $\mathbf{u}_i^0 = \mathbf{o}$, $\mathbf{u}_i^1 = [u_j^k; j \in I, k \in N, u_j^k \neq u_i^0]$, $\mathbf{u}_i^2 = [u_j^k; i \in I, k \in N, u_j^k \neq u_i^0, u_i^1] \dots$. Then $\mathbf{u}_i^0 > \mathbf{u}_i^1 > \mathbf{u}_i^2 > \dots$ is a chain of type ω^* . The convex sublattice generated by these chains will be denoted by C . Take the sublattice $A \cup C$, then A is a dual ideal and C is an ideal of this lattice. We adjoin further elements $\mathbf{w}_i^0, \mathbf{w}_i^1, \mathbf{w}_i^2, \dots$ and $\mathbf{z}_i^1, \mathbf{z}_i^2, \mathbf{z}_i^3 \dots$, which are defined as follows:

$$\mathbf{w}_i^1 = [\mathbf{u}_i^1, v_i], \mathbf{w}_i^2 = [\mathbf{u}_i^2, v_i], \mathbf{w}_i^3 = [\mathbf{u}_i^3, v_i] \dots$$

and

$$\mathbf{z}_i^1 = [\mathbf{u}_i^1, z_i^1], \mathbf{z}_i^2 = [\mathbf{u}_i^2, z_i^2], \mathbf{z}_i^3 = [\mathbf{u}_i^3, z_i^3] \dots$$

Then the join of the chains $\mathbf{u}_i^0 > \mathbf{u}_i^1 > \mathbf{u}_i^2 > \dots$ and $\mathbf{w}_i^0 > \mathbf{w}_i^1 > \mathbf{w}_i^2 > \dots$ form a sublattice isomorphic to $\mathfrak{C}_2 \times N^*$. For every j , $\mathbf{u}_i^j$, $\mathbf{z}_i^{j+1}$ and $\mathbf{w}_i^{j+1}$ generate $\mathfrak{M}_3$. For every $i \in I$ all these elements form a sublattice, the flap

$F_1^i = \{\mathbf{u}_i^0, \mathbf{u}_i^1, \mathbf{u}_i^2 \dots\} \cup \{\mathbf{w}_i^0, \mathbf{w}_i^1, \mathbf{w}_i^2 \dots\} \cup \{\mathbf{z}_i^1, \mathbf{z}_i^2, \mathbf{z}_i^3 \dots\}$ isomorphic to the lattice F_1 .

Similarly, we define for the elements v_{ij} the flaps

$F_2^{ij} = \{\mathbf{u}_{ij}^0, \mathbf{u}_{ij}^1, \mathbf{u}_{ij}^2 \dots\} \cup \{\mathbf{w}_{ij}^0, \mathbf{w}_{ij}^1, \mathbf{w}_{ij}^2 \dots\} \cup \{\mathbf{z}_{ij}^2, \mathbf{z}_{ij}^3, \mathbf{z}_{ij}^4 \dots\}$ isomorphic to F_2 .

Let M be $A \cup C \cup \bigcup (F_1^i, F_2^{ij} \mid i, j \in I)$.

M can be visualised as follows, see Figure 2.

It is easy to see that M is a sublattice of K . The lattice K is an arguesian lattice, consequently M is again arguesian. We prove that M is simple. We know that A and the F_1^i -s are simple lattices and the intervals $[\mathbf{u}_i^k, \mathbf{u}_i^{k+1}]$ and $[\mathbf{u}_j^k, \mathbf{u}_j^{k+1}]$ resp. $[\mathbf{u}_{ij}^k, \mathbf{u}_{ij}^{k+1}]$ and $[\mathbf{u}_j^k, \mathbf{u}_j^{k+1}]$ are projective in C . These imply that any two prime intervals are projective, which proves that M is a simple lattice.

M contains the chains $\mathbf{w}_i^1 > \mathbf{w}_i^2 > \mathbf{w}_i^3 > \dots$ and $\mathbf{w}_{ij}^0 > \mathbf{w}_{ij}^1 > \mathbf{w}_{ij}^2 \dots$, where $\mathbf{w}_i^1, \mathbf{w}_i^2, \dots$ resp. $\mathbf{w}_{ij}^1, \mathbf{w}_{ij}^2 \dots$, $(i, j \in I)$ are meet irreducible elements and M has no other chains of this type. Then for any automorphism the image of $\mathbf{w}_i^1$ must be $\mathbf{w}_j^1$ for some j and similarly the image of $\mathbf{u}_{ij}^1$ is some $\mathbf{u}_{k\ell}^1$. This yields that the restriction of an automorphism to the atoms of the dual ideal A of M is a permutation, where V and W are invariant. This proves that the automorphism group of M is isomorphic to $\mathfrak{G}$.

Figure 2

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