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On a sharp inequality for the Laplacian of a polyharmonic function

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Abstract. We give a short proof of the sharp inequality $|\Delta f(0)| \le 4d(m-1)^2 \sup_{\boldsymbol{B}} |f|$, where f is a polyharmonic function of order m on the unit ball in the Euclidean d-space.

Let $\boldsymbol{B}^d$ denote the unit ball in the Euclidean d-space, and let $H_m(\boldsymbol{B}^d)$ $(m = 1, 2, \dots)$ denote the class of those functions f on $\boldsymbol{B}^d$ for which $\Delta^m f = 0$, where Δ^m stands for the m-th power of the Laplacian. In [3] KOUNCHEV considered the inequality

$$|\Delta^k f(0)| \le C \sup_{\boldsymbol{B}^d} |f|, \quad f \in H_m(\boldsymbol{B}^d), \tag{1}$$

and stated that the best constant is $C = 2^k T_{m-1}^{(k)}(1)$, where $T_{m-1}^{(k)}$ is the k-th derivative of the Chebyshev polynomial T_{m-1}. However, elementary examples show that C must depend on d. By Theorem 1 below, the best constant in the case $k = 1$ is equal to $4d(m-1)^2$.

For two integers n and k, $1 \le k \le n$, let

$$A(n,k) = n^2(n^2 - 1^2) \dots (n^2 - (k-1)^2)$$

and

$$A(n,k,d) = A(n,k)d(d+2) \dots (d+2k-2)/(2k-1)!!\,.$$

Thus $A(n,k) = A(n,k,1)$.

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Theorem 1. *If $1 \le k \le m-1$ and $d \ge 1$, then the best constant in* (1) *is $C = 4^k A(m-1,k,d)$.*

We will deduce this theorem from the theorem of the Markov brothers [1, p. 323], which we state as follows:

Theorem M. *Let $P_n(t)$ be a polynomial of degree n such that $|P_n(t)| \le 1$ for all $0 \le t \le 1$. Then*

$$|P_n^{(k)}(0)| \le 2^k T_n^{(k)}(1) \tag{2}$$

for $1 \le k \le n$.

Recall that $T_n(t) = \cos(n \arccos t)$ and

$$T_n^{(k)}(1) = A(n,k)/(2k-1)!!\,.$$

Corollary. *Let $Q(t) = P_n(t^2)$, where P_n is as above. Then*

$$|Q^{2k}(0)| \le 4^k A(n,k).$$

Proof. This follows from Theorem M by using the formula

$$Q^{(2k)}(0) = (2k)!\,(k!)^{-1} P_n^{(k)}(0) = 2^k (2k-1)!!\, P_n^{(k)}(0). \qquad \square$$

Since $T_{2n}(t) = (-1)^n T_n(1-2t^2)$, we see that

$$|T_{2n}^{(2k)}(0)| = 4^k A(n,k), \tag{3}$$

and therefore the corollary proves a particular case of Theorem 1. In the general case we need a simple formula as well.

Lemma. *Let $u_j(x) = |x|^{2j}$, $x \in \boldsymbol{B}^d$. Then $\Delta^k u_j(0) = 0$ for $k \ne j$ and*

$$\Delta^k u_k(0) = (2k)!!\, d(d+2)\dots(d+2k-2).$$

Proof. By direct computation one shows that

$$\Delta u_j(x) = 2j(2j-2+2d)|x|^{2j-2}.$$

Successive applications of this formula yield the conclusion of the lemma. $\square$

Proof of Theorem 1. Let $f \in H_m(\boldsymbol{B}^d)$ and $|f(x)| \le 1$ for all $x \in \boldsymbol{B}^d$. Let u denote the radialization of f,

$$u(x) = \int f(Gx)dG, \tag{4}$$

where the integral is taken over the group of all orthogonal transformations of the d-space. Since the Laplacian commutes with the orthogonal transformations, we have that $u \in H_m(\boldsymbol{B}^d)$, $\Delta^k u(0) = \Delta^k f(0)$ as well as $|u(x)| \le 1$ for all $x \in \boldsymbol{B}^d$. And since u is a radial function, there is a polynomial $P_{m-1}(t)$, $\deg P_{m-1} \le m-1$, such that $u(x) = P_{m-1}(|x|^2)$. (This can be proved by induction or by using the Almansi theorem [2]). By using the lemma we find that

$$\Delta^k u(0) = Q^{(2k)}(0)d(d+2)\dots(d+2k-2)/(2k-1)!!, \tag{5}$$

where $Q(t) = P_{m-1}(t^2)$. Now we apply the Corollary to Theorem M to obtain

$$|\Delta^k f(0)| = |\Delta^k u(0)| \le 4^k A(m-1,k,d).$$

Finally, to show that the constant is the best possible, let $f(x) = T_{m-1}(1-2|x|^2) = (-1)^m T_{2m-2}(|x|)$. Using the formulas (5) ($u = f$) and (3) shows that $\Delta^k f(0) = 4^k A(m-1,k,d)$, and this completes the proof. $\square$

Remark 1. The best constant can be attained on non-radial functions. As an example consider the case where $d=2$, $m=2$ and $k=1$, and identify the 2-space with the complex plane. Let

$$f(x) = -1 + 2(1-2|x|^2)u(x),$$
$$u(x) = \Re((1-ax^2)^{-1})$$

for some a, $|a| \le 1$. Since $0 < u(x) \le (1-|x|^2)^{-1}$ we have $-1 \le f(x) \le 1$ for all $x \in \boldsymbol{B}^2$, and $-\Delta f(0) = 8u(0) = 8 = 4^k A(m-1,k,d)$.

Remark 2. A slight improvement of Theorem 1 follows from the proof. Let

$$I(r,f) = \int_S f(ry)d\sigma(y), \quad 0 < r < 1,$$

where $d\sigma$ is the normalized surface measure on S. Then there holds the sharp inequality

$$(6) \qquad |\Delta^k f(0)| \le 4^k A(m-1,k,d) \sup_{r<1} |I(r,f)|, \quad f \in H_m(\boldsymbol{B}^d).$$

It should be noted, however, that (6) is obtained by an application of (1) to the function u defined by (4).

Remark 3. Only minor modifications of the proof of Theorem 1 are needed to show the following: Let $\alpha > 0$, $1 \le q \le \infty$ and let $C_q = C_q(n,k,\alpha)$ denote the best constant in the inequality

$$|P_n^{(k)}(0)| \le C_q \left(\int_0^1 |P_n(t)|^q t^{\alpha-1} dt \right)^{1/q}.$$

Then there holds the sharp inequality

$$|\Delta^k f(0)| \le D_q(m-1,k,d/2) \left(\int_{\boldsymbol{B}^d} |f|^q d\nu \right)^{1/q},$$

for $f \in H_m(\boldsymbol{B}^d)$, where $d\nu$ is the normalized measure on $\boldsymbol{B}^d$ and

$$D_n(n,k,\alpha) = (2/d)^{1/q}\, 2^k d(d+2) \dots (d+2k-2)\, C_q(n,k,\alpha).$$

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