

Starlikeness properties of meromorphic m -valent functions

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Abstract. We find conditions regarding the values of β such that $F(z)$ given by (*) is nonvanishing, when $f \in \Sigma_{m,n}^*(\alpha)$ with $n \geq -m + 1$ and employ a differential subordination technique, for certain conditions on β , to obtain the sharpest value of $\delta = \delta(\alpha, \beta, \gamma, m, n)$ such that

$$I_{\beta, \gamma}^m (\Sigma_{m,n}^*(\alpha)) \subset \Sigma_{m,n}^*(\delta) \subset \Sigma_{m,n}^*(\alpha).$$

1. Introduction

Let $\Sigma_{m,n}$ be the class of meromorphic m -valent functions f defined in $\dot{U} = \{z \in \mathbb{C}, 0 < |z| < 1\}$ of the form

$$f(z) = \frac{1}{z^m} + a_n z^n + a_{n+1} z^{n+1} + \dots,$$

where m and n are integers such that $m > 0$ and $n \geq -m + 1$.

A function $f \in \Sigma_{m,n}$ is called starlike of order α , $0 \leq \alpha < m$ if

$$\operatorname{Re} \left(\frac{-z f'(z)}{f(z)} \right) > \alpha.$$

Let $\Sigma_{m,n}^*(\alpha)$ be the class of all such functions.

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Let γ, β be real numbers, $\gamma > 0$ and let us define the integral operator $I_{\gamma,\beta}^m : \Sigma_{m,n}^*(\alpha) \rightarrow \Sigma_{m,n}$,

$$(*) \quad I_{\gamma,\beta}^m(f)(z) = F(z) = \left(\frac{\gamma}{z^{\gamma+m\beta}} \int_0^z t^{\gamma+m\beta-1} f^\beta(t) dt \right)^{1/\beta}, \quad z \in \dot{U}.$$

In [1], [2] and [3] HASSON AL-AMIRI and PETRU T. MOCANU have investigated the properties of functions f belonging to $\Sigma_{1,n}^*(0) = \Sigma_n^*$ for $n > 0$. In [4] M.K. AOUF and M. HOSSEN obtained some interesting results concerning the function f from $\Sigma_{m,n}$ with $n \geq 0$.

In this paper we extend a part of results of H. AL-AMIRI and P.T. MOCANU for f belonging to $\Sigma_{m,n}^*(0) = \Sigma_{m,n}^*$ with $n > 0$ and other results for f belonging to $\Sigma_{m,n}^*$ with $n \geq -m + 1$.

Moreover, we find conditions regarding the values of β such that $F(z)$ given by (*) is nonvanishing, when $f \in \Sigma_{m,n}^*(\alpha)$ with $n \geq -m + 1$ and employ a differential subordination technique, for certain conditions on β , to obtain the sharpest value of $\delta = \delta(\alpha, \beta, \gamma, m, n)$ such that

$$I_{\beta,\gamma}^m(\Sigma_{m,n}^*(\alpha)) \subset \Sigma_{m,n}^*(\delta) \subset \Sigma_{m,n}^*(\alpha).$$

In the case of holomorphic m -valent functions similar results were obtained by PONNUSAMY [5].

2. Preliminaries

We will need the following definitions and lemmas to prove our main results.

If f and g are holomorphic functions in the unit disc U and if g is univalent, then we say that f is subordinate to g , written $f \prec g$ or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$. The holomorphic function h , with $h(0) = 0$ and $h'(0) \neq 0$ is starlike in U (i.e. h is univalent in U and $h(U)$ is starlike with respect to the origin) if and only if $\operatorname{Re}[zh'(z)/h(z)] > 0$ for $z \in U$.

The holomorphic function h in U is convex (i.e. h is univalent and $h(U)$ is convex domain) if $f'(0) \neq 0$ and

$$\operatorname{Re} \frac{zh''(z)}{h'(z)} + 1 > 0, \quad \text{for } z \in U.$$

Lemma A [2]. Let n be a positive integer and α be real, with $0 < \alpha < n$. Let q be holomorphic in U , with $q(0) = 0$, $q'(0) \neq 0$ and suppose that

$$\text{i) } \operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}.$$

Let

$$\text{ii) } h(z) = nzq'(z) - \alpha q(z).$$

If $p(z) = p_n z^n + \dots$ is holomorphic in U and $zp'(z) - \alpha p(z) \prec h(z)$, then $p(z) \prec q(z)$ and this result is sharp.

Lemma B [3]. Let n be a positive integer, let $\lambda > 0$ and let $\beta^* = \beta^*(\lambda, n)$ be the solution of the equation $\beta\pi - 2 \operatorname{arctg}(n\lambda\beta) = 0$. Let $\alpha = \alpha(\beta, \lambda, n) = \frac{2}{\pi} \operatorname{arctg}(n\lambda\beta) - \beta$ for $0 < \beta \leq \beta^*$.

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is analytic in U , then

$$p(z) - \lambda zp'(z) \prec \left(\frac{1+z}{1-z} \right)^\alpha$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z} \right)^\beta.$$

Lemma C [3]. Let n be a positive integer, $\lambda > 0$, $\beta^* = \beta^*(\lambda, n)$ be the solution of the equation $\beta\pi = \frac{3\pi}{2} - \operatorname{arctg}(n\lambda\beta)$. Let $\alpha = \alpha(\beta, \lambda, n) = \beta + \frac{2}{\pi} \operatorname{arctg}(n\lambda\beta)$ for $0 < \beta < \beta^*$. If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is analytic in U , then

$$p(z) + \lambda zp'(z) \prec \left(\frac{1+z}{1-z} \right)^\alpha$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z} \right)^\beta.$$

Lemma D [6]. Let h be starlike in U and let $p(z) = 1 + p_n z^n + \dots$ be holomorphic in U . If $\frac{zp'(z)}{p(z)} \prec h(z)$, then $p \prec q$, where $q(z) = \exp \frac{1}{n} \int_0^z \frac{h(t)}{t} dt$.

Lemma E [7]. Let $\mu(t)$ be a positive measure on the unit interval $I = [0, 1]$. Let $g(t, z)$ be a complex valued function defined on $U \times [0, 1]$ and integrable in t for each $z \in U$ and for almost all $t \in [0, 1]$, and suppose that $\operatorname{Re}\{g(t, z)\} > 0$ on U and $g(z) = \int_I g(t, z) d\mu(t)$. If for fixed λ ($0 \leq \lambda < 2\pi$) $g(t, re^{i\lambda})$ is real for r real and

$$\operatorname{Re} \left\{ \frac{1}{g(t, z)} \right\} \geq \frac{1}{g(t, re^{i\lambda})},$$

for $|z| \leq r$ and $t \in [0, 1]$, then

$$\operatorname{Re} \left\{ \frac{1}{g(z)} \right\} \geq \frac{1}{g(re^{i\lambda})}$$

for $|z| \leq r$ and $0 \leq \lambda < 2\pi$.

This Lemma can be proved in a similar manner as that of Lemma 2 of WILKEN and FENG in [7].

Theorem A [8]. Let β', γ' be complex numbers with $\beta' \neq 0$ and let k be a positive integer. Let h be analytic in U with $h(0) = a \neq 0$, $\operatorname{Re}[\beta'a + \gamma'] > 0$, and

i) $\operatorname{Re}(\beta'h(z) + \gamma') > 0$.

If q is the analytic solution of the Briot-Bouquet differential equation

$$q(z) + \frac{kzq'(z)}{\beta'q(z) + \gamma'} = h(z)$$

given by

$$q(z) = \left(\frac{\beta'}{k} \int_0^z [H(tz)/H(z)]^{\frac{\beta'a}{k}} t^{\frac{\gamma'}{k}-1} dt \right)^{-1} - \frac{\gamma'}{\beta'}$$

with

$$H(z) = z \exp \int_0^t [h(t) - a]/at dt$$

and if

ii) h is convex or $Q(z) = \frac{zq'(z)}{\beta'q(z) + \gamma'}$ is starlike then q and h are univalent. Furthermore, if the holomorphic function $p(z) = a + a_k z^k + \dots$ satisfies

$$p(z) + \frac{zp'(z)}{\beta'p(z) + \gamma'} \prec h(z)$$

then $p \prec q \prec h$ and q is the best (a, k) dominant.

More general form of this theorem may be found in [8].

We also need the following well known formula.

For a, b, c real numbers with $c \neq 0, -1, -2, \dots$ the function

$$(**) \quad {}_2F_1(a, b, c; z) = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots, \quad z \in U,$$

is called the (Gaussian) hypergeometric function.

If $c > a > 0$, then

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-zt)^{-b} dt$$

and

$${}_2F_1(a, b, c; z) = (1-z)^{-b} {}_2F_1\left(b, c-a, c, \frac{z}{z-1}\right).$$

3. Main results

Theorem 1. Let q be holomorphic, with $q(0) = 0$, $q'(0) \neq 0$ and suppose that $\operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 > \frac{m}{n+m}$. Let $h(z) = (m+n)zq'(z) - mq(z)$.

If n is positive integer, $f \in \Sigma_{m,n}$ and

$$z^{m+1}f'(z) + m \prec h(z),$$

then $z^m f(z) - 1 \prec q(z)$ and the result is sharp.

PROOF. If we denote $p(z) = z^m f(z) - 1$, then $p(z) = a_n z^{m+n} + \dots$ is holomorphic in U and

$$(1) \quad zp'(z) - mp(z) = z^{m+1}f'(z) + m \prec h(z).$$

Applying Lemma A with $\alpha = m$, $n = m+n$ from (1) we obtain that $z^m f(z) - 1 \prec q(z)$.

Corollary 1. Let M be a positive real number and n, m be positive integers. If $f \in \Sigma_{m,n}$ and

$$(2) \quad |z^{m+1}f'(z) + m| < M,$$

then

$$(3) \quad |z^m f(z) - 1| < \frac{M}{n}.$$

PROOF. If we take $q(z) = \frac{Mz}{n}$, then $h(z) = Mz$ and the result follows from Theorem 1.

Corollary 1. *Let $f \in \Sigma_{m,n}$ with m, n positive integers and suppose that*

$$(4) \quad |z^{m+1} f'(z) + m| < \frac{mn}{\sqrt{m^2 + n^2}},$$

then $f \in \Sigma_{m,n}^*$.

PROOF. Let $0 < M \leq \frac{mn}{\sqrt{m^2 + n^2}}$. If we denote $P(z) = z^m f(z)$, then by Corollary 1 we obtain $|P(z) - 1| < \frac{M}{n}$, which implies that $P(z) \neq 0$ in U . Hence the function $p(z) = -\frac{zf'(z)}{f(z)} = m + p_n z^{m+n} + \dots$ is holomorphic in U and we have $-z^{m+1} f'(z) = P(z)p(z)$. Hence the inequality (4) becomes

$$(5) \quad |P(z)p(z) - m| < M, \quad z \in U.$$

We have to show that this inequality implies $\operatorname{Re} p(z) > 0$ in U . Suppose $\operatorname{Re} p(z) \not> 0$ in U . Then there exists a point $z_0 \in U$ such that $p(z_0) = is$ where s is real number.

If $M \leq \frac{mn}{\sqrt{m^2 + n^2}}$, then $|\operatorname{Im} P(z)| < \frac{\sqrt{m^2 - M^2}}{m} |P(z)|$ and

$$|P(z)is - m|^2 - M^2 = |P(z)|^2 s^2 + 2ms \operatorname{Im} P(z) + m^2 - M^2 \geq 0$$

for all real s , then we deduce that

$$|P(z)is - m|^2 \geq M^2$$

for $z \in U$ and s real.

In particular $|P(z_0)p(z_0) - m| \geq M$ which contradicts (5).

Hence we must have $\operatorname{Re} p(z) > 0$ in U , which shows that $f \in \Sigma_{m,n}^*$.

Remark. For $m = 1$ Theorem 1, Corollary 1, Corollary 2 were proved in [2].

Theorem 2. *Let m, n be integers such that $m > 0$ and $n \geq -m + 1$.
Let*

$$(6) \quad \beta^* = \beta^*(m, n)$$

be the root of the equation $\beta\pi - 2 \operatorname{arctg} \left(\frac{n+m}{m} \beta \right) = 0$.

Let

$$(7) \quad \alpha = \alpha(m, n, \beta) = \frac{2}{\pi} \operatorname{arctg} \frac{(n+m)\beta}{n} - \beta$$

for $0 < \beta < \beta^$.*

If $f \in \Sigma_{m,n}$ and

$$(8) \quad |\arg[-z^{m+1}f'(z)]| < \alpha \frac{\pi}{2}, \quad z \in U,$$

then

$$(9) \quad |\arg[z^m f(z)]| < \beta \frac{\pi}{2}.$$

PROOF. If we let $p(z) = z^m f(z) = 1 + a_n z^{m+n} + \dots$, then

$$\frac{-z^{m+1}f'(z)}{m} = p(z) - \frac{1}{m}zp'(z)$$

and the inequality (8) becomes

$$p(z) - \frac{1}{m}zp'(z) \prec \left(\frac{1+z}{1-z} \right)^\alpha.$$

Hence by Lemma B with $\lambda = \frac{1}{m}$ and $m+n \geq 1$ we deduce

$$p(z) \prec \left(\frac{1+z}{1-z} \right)^\beta$$

where α and β satisfy (6) and (7).

Theorem 3. Let m, n be integers such that $m > 0$ and $n \geq -m + 1$ and suppose that α, β satisfy (6) and (8).

If $f \in \Sigma_{m,n}$ and $|\arg[-z^{m+1}f'(z)]| < \alpha\frac{\pi}{2}$, then

$$\left| \arg \left(-\frac{zf'(z)}{f(z)} \right) \right| < \operatorname{arctg} \frac{m+n}{m} \beta, \quad z \in U,$$

hence $f \in \Sigma_{m,n}^*$.

PROOF. Applying Theorem 1 from (9) and (10) we deduce

$$\left| \arg \left(-\frac{zf'(z)}{f(z)} \right) \right| \leq |\arg[-z^{m+1}f'(z)]| + |\arg[z^m f(z)]| < \gamma\frac{\pi}{2},$$

where

$$\gamma = \beta + \alpha = \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{m} \beta.$$

Consequently

$$\left| \arg \left(-\frac{zf'(z)}{f(z)} \right) \right| < \operatorname{arctg} \frac{m+n}{n} \beta < \frac{\pi}{2}$$

from which it follows that

$$\operatorname{Re} \left(-\frac{zf'(z)}{f(z)} \right) > 0.$$

Theorem 4. Let m, n be integers such that $m > 0$ and $n \geq -m + 1$. Suppose that the positive real numbers α, β, γ, c satisfy

$$(10) \quad \alpha = \beta + \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{c} \beta$$

$$(11) \quad \beta = \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{m} \gamma - \gamma.$$

If $f \in \Sigma_{m,n}$ and

$$(12) \quad |\arg[-z^{m+1}f'(z)]| < \alpha\frac{\pi}{2}, \quad z \in U,$$

then

$$\left| \arg \left(-\frac{zF'(z)}{F(z)} \right) \right| < \operatorname{arctg} \left(\frac{m+n}{m} \gamma \right), \quad z \in \dot{U},$$

consequently $F \in \Sigma_{m,n}^*$, where

$$(13) \quad F(z) = \frac{c}{z^{c+m}} \int_0^z t^{c+m-1} f(t) dt, \quad z \in \dot{U}.$$

PROOF. From (13) we obtain

$$(c+m)F(z) + zF'(z) = cf(z),$$

and if we denote $p(z) = \frac{-z^{m+1}F'(z)}{m} = 1 + p_m z^{m+n} + \dots$, then

$$p(z) + \frac{1}{c}zp'(z) = -\frac{1}{m}z^{m+1}f'(z),$$

and (12) becomes

$$p(z) + \frac{1}{c}zp'(z) \prec \left(\frac{1+z}{1-z} \right)^\alpha.$$

Applying Lemma C we obtain

$$p(z) \prec \left(\frac{1+z}{1-z} \right)^\beta$$

that means

$$(14) \quad |\arg[-z^{m+1}F'(z)]| < \beta \frac{\pi}{2},$$

where α and β satisfy

$$\alpha = \beta + \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{c} \beta.$$

Now by Theorem 2, the inequality (14) implies

$$|\arg[z^m F(z)]| < \gamma \frac{\pi}{2},$$

where

$$\beta + \gamma = \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{m} \gamma$$

and we deduce

$$\left| \arg \left(-\frac{zF'(z)}{F(z)} \right) \right| \leq |\arg[-z^{m+1}F'(z)]| + |\arg[z^m F(z)]|$$

$$< (\beta + \gamma) \frac{\pi}{2} = \operatorname{arctg} \frac{m+n}{m} \gamma < \frac{\pi}{2}.$$

Theorem 5. *Let m, n be integers, $m > 0$ and $n \geq -m + 1$. Let $\alpha, \beta, \gamma, \beta_1$ be real numbers so that $\alpha \in [0, m)$, $\gamma > 0$*

$$\text{i) } 1 = \beta_1 + \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{\gamma} \beta_1$$

$$\text{ii) } \beta_1 < |\beta| < \frac{m+n}{2(m-\alpha)}.$$

Let $F : U \rightarrow \mathbb{C}$ defined by (). If $f \in \Sigma_{m,n}^*(\alpha)$, then $\operatorname{Re}[z^m F(z)] > 0$, consequently $F(z) \neq 0$ in $\dot{U}$.*

PROOF. Since $f \in \Sigma_{m,n}^*(\alpha)$ we have $f(z) \neq 0$, for all $z \in U$ and, let $\varphi(z) = z^m f(z)$.

$$\frac{z\varphi'(z)}{\varphi(z)} = m + \frac{zf'(z)}{f(z)} \prec m - m \frac{1 + (1 - \frac{2\alpha}{m})z}{1 - z} = \frac{2(\alpha - m)z}{1 - z} = h(z).$$

Applying Lemma D we obtain that

$$(15) \quad \varphi(z) = z^m f(z) \prec (1 - z)^{\frac{2(\alpha-m)}{m+n}}.$$

This implies that $z^m f(z) \neq 0$ in U .

Let $g(z) = (z^m F(z))^\beta = 1 + g_n z^{m+n} + \dots$. From (1) we obtain that

$$g(z)z^\gamma = \gamma \int_0^z f^\beta(t) t^{\gamma+m\beta-1} dt,$$

this implies that

$$(16) \quad g(z) + \frac{1}{\gamma} z g'(z) = (z^m f(z))^\beta \prec (1 - z)^{\frac{2(\alpha-m)\beta}{m+n}}.$$

We observe that $|\arg(1 - z)| < \frac{\pi}{2}$, when $z \in U$. From ii) and (16) it follows that

$$\operatorname{Re} \left(g(z) + \frac{1}{\gamma} z g'(z) \right) > 0,$$

consequently

$$g(z) + \frac{1}{\gamma} z g'(z) \prec \frac{1+z}{1-z}.$$

Applying Lemma C with $\alpha = 1$, $n = m + n$ we obtain that

$$(17) \quad g(z) \prec \left(\frac{1-z}{1+z} \right)^{\beta_1},$$

where β_1 is given by i). We observe that

$$0 < 1 - \beta_1 = \frac{2}{\pi} \arctg \frac{n+m}{\gamma} \beta_1 < 1,$$

i.e. $\beta_1 \in (0, 1)$. From (17) it follows that

$$|\arg[z^m F(z)]^\beta| < \beta_1 \frac{\pi}{2},$$

that implies

$$(18) \quad |\arg[z^m F(z)]| < \frac{\beta_1}{|\beta|} \frac{\pi}{2}.$$

Taking into account that $\beta_1 < |\beta|$, from (18) we deduce that $\operatorname{Re}(z^m F(z)) > 0$ in $\dot{U}$, and this implies $F(z) \neq 0$ in $\dot{U}$.

Theorem 6. Let $m, n, \alpha, \beta, \gamma, \beta_1$ be real numbers satisfying all the conditions of Theorem 5 and suppose that $\beta < 0$.

If $f \in \Sigma_{m,n}^*(\alpha)$ and $F(z)$ is given by (1), then

$$(19) \quad F \in \Sigma_{m,n}^*(\delta),$$

where

$$\delta = -\frac{\gamma}{\beta} \frac{1}{{}_2F_1\left(\frac{2\beta(\alpha-m)}{m+n}, 1, \frac{\gamma}{m+n} + 1; \frac{1}{2}\right)} + \frac{\gamma + \beta m}{\beta},$$

${}_2F_1$ given by (**), and this result is sharp.

PROOF. From Theorem 1 we obtain that if $f \in \Sigma_{m,n}^*(\alpha)$, then $F(z) \neq 0$ hence

$$p(z) = -\frac{zF'(z)}{F(z)} = m + c_n z^{m+n} + \dots$$

is an analytic function in U with $p(0) = m$. From (1) we easily obtain that

$$(20) \quad p(z) + \frac{zp'(z)}{-\beta p(z) + \gamma + m\beta} = -\frac{zf'(z)}{f(z)}, \quad z \in U.$$

The condition $f \in \Sigma_{m,n}^*(\alpha)$ implies that

$$(21) \quad -\frac{zf'(z)}{f(z)} \prec m \frac{1 + \left(1 - \frac{2\alpha}{m}\right)z}{1 - z} = h(z).$$

From (20) and (21) we obtain that

$$p(z) + \frac{zp'(z)}{-\beta p(z) + \gamma + m\beta} \prec h(z) = m \frac{1 + \left(1 - \frac{2\alpha}{m}\right)z}{1 - z}.$$

It is easy to show that h is a convex function in U and for $\beta < 0$, $\gamma > 0$, $\operatorname{Re}(-\beta h(z) + \gamma) > 0$. Hence we can apply Theorem A with $\beta' = -\beta$, $\gamma' = \gamma + \beta m$, $a = m$, $k = m + n$. Observe that $\operatorname{Re}(\beta' a + \gamma') = \gamma > 0$ and we obtain that

$$(22) \quad p(z) \prec q(z) \prec h(z)$$

and this result is sharp, where

$$(23) \quad \begin{aligned} q(z) &= \left(\frac{-\beta}{m+n} \int_0^z t^{\frac{-\beta m}{m+n}} \left(\frac{1-tz}{1-z} \right)^{\frac{-2\beta(\alpha-m)}{m+n}} t^{\frac{\gamma+\beta m}{m+n}-1} dt \right)^{-1} + \frac{\beta m + \gamma}{\beta} \\ &= \left(\frac{-\beta}{m+n} \int_0^z t^{\frac{\gamma}{m+n}-1} \left(\frac{1-tz}{1-z} \right)^{\frac{-2\beta(\alpha-m)}{m+n}} dt \right)^{-1} + \frac{\beta m + \gamma}{\beta}. \end{aligned}$$

If we consider

$$Q(z) = \int_0^1 t^{\frac{\gamma}{m+n}-1} \left(\frac{1-tz}{1-z} \right)^{\frac{-2\beta(\alpha-m)}{m+n}} dt$$

and we denote with

$$b = \frac{2\beta(\alpha-m)}{m+n}, \quad a = \frac{\gamma}{m+n}, \quad c = a + 1 = \frac{\gamma}{m+n} + 1,$$

we observe that $c > a > 0$ and

$$\begin{aligned}
 Q(z) &= (1-z)^b \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} {}_2F_1(a, b, c, z) \\
 (24) \quad &= (1-z)^b \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} {}_2F_1\left(b, c-a, c, \frac{z}{z-1}\right) (1-z)^{-b} \\
 &= \int_0^1 \frac{1-z}{1-(1-t)z} d\mu(t)
 \end{aligned}$$

where

$$d\mu(t) = t^{b-1}(1-t)^{c-b-1} \frac{\Gamma(a)}{\Gamma(c-b)\Gamma(b)} dt.$$

Let $g(z, t) = \frac{1-z}{1-(1-t)z}$, we observe that $\operatorname{Re}\{g(t, z)\} > 0$, $|z| \leq r < 1$, $g(t, -r) \in \mathbb{R}$, $0 \leq r < 1$, $t \in [0, 1]$ and

$$\operatorname{Re} \left\{ \frac{1}{g(z, t)} \right\} = \operatorname{Re} \left\{ \frac{1-(1-t)z}{1-z} \right\} \geq \frac{1+(1-t)r}{1+r} = \frac{1}{g(t, -r)}$$

for $|z| \leq r < 1$ and $t \in [0, 1]$. Therefore, by using Lemma E we deduce that

$$\operatorname{Re} \left\{ \frac{1}{Q(z)} \right\} \geq \frac{1}{Q(-r)}, \quad |z| \leq r < 1,$$

and by letting $r \rightarrow 1^-$, we obtain

$$\operatorname{Re} \left\{ \frac{1}{Q(z)} \right\} \geq \frac{1}{Q(-1)}, \quad z \in U.$$

Taking into account that $\beta < 0$ we obtain that

$$\begin{aligned}
 \operatorname{Re} q(z) &= \operatorname{Re} \left(\frac{m+n}{-\beta Q(z)} + \frac{\beta m + \gamma}{\beta} \right) \geq \frac{m+n}{-\beta Q(-1)} + \frac{\gamma + \beta m}{\beta} \\
 (25) \quad &= -\frac{\gamma}{\beta} \frac{1}{{}_2F_1\left(\frac{2\beta(\alpha-m)}{m+n}, 1, \frac{\gamma}{m+n} + 1, \frac{1}{2}\right)} + \frac{\gamma + \beta m}{\beta} = \delta.
 \end{aligned}$$

From (22) and (24) it follows that

$$\operatorname{Re} \left(\frac{-zF'(z)}{F(z)} \right) > \min_{|z|<1} \operatorname{Re} q(z) = \delta,$$

consequently $F \in \Sigma_{m,n}^*(\delta)$ and this result is sharp.

References

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