

On Nemytskii operator for set-valued functions

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Abstract. This paper is devoted to Nemytskii operator mapping some subfamily of the space of class C^1 functions into a space of set-valued functions, and can be considered as generalization of [13].

J. MATKOWSKI has proved that every function h , which generates the Nemytskii operator mapping the Banach space of Lipschitzian functions into itself and satisfying the Lipschitz condition, must be Jensen function with respect to the second variable (see [3], [4], [5]). His result was generalized by A. SMAJDOR and W. SMAJDOR for set-valued functions ([10], [11]). We will restrict our attention to the Nemytskii operator defined on some subspace of the space considered by Matkowski.

We will start with some definitions and notations. Let $(X, \|\cdot\|)$ be a real normed space and let $C \subseteq X$ be nonempty. Let $f : \langle a, b \rangle \rightarrow X$, and $x_1, \dots, x_p \in \langle a, b \rangle$ be distinct points. The *divided difference* $[x_1, \dots, x_p; f]$ of f at points $x_1, \dots, x_p$ is defined by recurrence

$$\begin{aligned} [x_1, f] &= f(x_1), \\ [x_1, \dots, x_p; f] &= \frac{[x_2, \dots, x_p; f] - [x_1, \dots, x_{p-1}; f]}{x_p - x_1}, \quad p \geq 2 \end{aligned}$$

(cf. [2]).

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Let us consider

$$\text{lip}^2(\langle a, b \rangle, C) := \left\{ f : \langle a, b \rangle \rightarrow C : \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; f]\| < \infty \right\}.$$

It is clear that $\text{lip}^2(\langle a, b \rangle, X)$ is a real normed space with the norm

$$\|f\| := \|f(a)\| + \|f(b)\| + \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; f]\|.$$

It is a Banach space, if the space $(X, \|\cdot\|)$ is complete.

Indeed, let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\text{lip}^2(\langle a, b \rangle, X)$ and let $\varepsilon > 0$ be fixed. Then,

$$\begin{aligned} \|f_n(a) - f_m(a)\| &\leq \varepsilon, \\ \|f_n(b) - f_m(b)\| &\leq \varepsilon, \end{aligned}$$

and

$$(1) \quad \frac{1}{|z - x|} \left\| \frac{f_n(z) - f_m(z) - f_n(y) + f_m(y)}{z - y} - \frac{f_n(y) - f_m(y) - f_n(x) + f_m(x)}{y - x} \right\| \leq \varepsilon,$$

for all $x, y, z \in \langle a, b \rangle$, distinct and $n, m \geq n_0$, where $n_0 = n(\varepsilon)$ is some number.

Let us fix $x \in (a, b)$. By (1), for $n, m \geq n_0$,

$$\begin{aligned} &\left\| \frac{f_n(x) - f_m(x) - f_n(b) + f_m(b)}{x - b} \right\| \\ &\leq \varepsilon(b - a) + \frac{\|f_n(b) - f_m(b) - f_n(a) + f_m(a)\|}{b - a} \leq \varepsilon(b - a) + \frac{2\varepsilon}{b - a}. \end{aligned}$$

Hence,

$$\begin{aligned} \|f_n(x) - f_m(x)\| &\leq \varepsilon((b - a)^2 + 2) + \|f_n(b) - f_m(b)\| \leq \varepsilon((b - a)^2 + 3), \\ n, m &\geq n_0. \end{aligned}$$

So, $\{f_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence, for each $x \in \langle a, b \rangle$. Define $f : \langle a, b \rangle \rightarrow X$ as follows $f(x) := \lim_{n \rightarrow \infty} f_n(x)$, $x \in \langle a, b \rangle$. By the continuity of the norm and (1),

$$\|[x, y, z; f_n - f]\| \leq \varepsilon, \quad n \geq n_0,$$

thus $f \in \text{lip}^2(\langle a, b \rangle, X)$, and $\|f_n - f\| \leq 3\varepsilon$, $n \geq n_0$.

The space $\text{lip}^2(\langle a, b \rangle, C) \subseteq \text{lip}^2(\langle a, b \rangle, X)$ can be considered as a metric space with a metric generated by the norm from $\text{lip}^2(\langle a, b \rangle, X)$.

First, we prove a few properties, which will be useful.

Let us consider the family

$$\text{lip}^1(\langle a, b \rangle, \mathbb{R}) := \left\{ f : \langle a, b \rangle \rightarrow \mathbb{R} : \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f]| < \infty \right\}.$$

By [2] (Lemma 1, p. 390), $\text{lip}^2(\langle a, b \rangle, \mathbb{R}) \subset \text{lip}^1(\langle a, b \rangle, \mathbb{R})$, but not conversely (for example the function $f(x) = x^{\frac{3}{2}}$, $x \in \langle 0, 1 \rangle$ belongs to the class $\text{lip}^1(\langle 0, 1 \rangle, \mathbb{R})$, but $f \notin \text{lip}^2(\langle 0, 1 \rangle, \mathbb{R})$).

It is easy to check that $(\text{lip}^1(\langle a, b \rangle, \mathbb{R}), \|\cdot\|)$ is a real Banach space with the norm

$$\|f\| := |f(a)| + \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f]|.$$

Lemma 1 (cf. the proof of Th. 1 in [2] p. 391). *If $f \in \text{lip}^2(\langle a, b \rangle, \mathbb{R})$, then f is of class C^1 in $\langle a, b \rangle$, and its derivative f' belongs to the class $\text{lip}^1(\langle a, b \rangle, \mathbb{R})$ and*

$$\sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f']| \leq 2 \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; f]|.$$

If a function $f : \langle a, b \rangle \rightarrow \mathbb{R}$ is differentiable in $\langle a, b \rangle$, and $f' \in \text{lip}^1(\langle a, b \rangle, \mathbb{R})$, then $f \in \text{lip}^2(\langle a, b \rangle, \mathbb{R})$, and

$$\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; f]| \leq \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f']|.$$

PROOF. Let $f \in \text{lip}^2(\langle a, b \rangle, \mathbb{R})$ and

$$M := \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; f]|.$$

Let $x_1, x_2, y_1, y_2 \in \langle a, b \rangle$ be fixed distinct points. Then

$$\begin{aligned} \left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{f(y_1) - f(y_2)}{y_1 - y_2} \right| &\leq \left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{f(x_2) - f(y_1)}{x_2 - y_1} \right| \\ &+ \left| \frac{f(x_2) - f(y_1)}{x_2 - y_1} - \frac{f(y_1) - f(y_2)}{y_1 - y_2} \right| \leq M(|x_1 - y_1| + |x_2 - y_2|). \end{aligned}$$

Thus, for a fixed point $x \in \langle a, b \rangle$, there exists the limit

$$\lim_{\substack{x_1, x_2 \rightarrow x \\ x_1 \neq x_2}} \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(x).$$

So, the function f is differentiable at every point $x \in \langle a, b \rangle$. Moreover, letting $x_2 \rightarrow x_1$, and $y_2 \rightarrow y_1$, we obtain that

$$|f'(x_1) - f'(y_1)| \leq 2M|x_1 - y_1| \quad \text{for all } x_1, y_1 \in \langle a, b \rangle,$$

which proves, that f is of class C^1 , its derivative $f' \in \text{lip}^1(\langle a, b \rangle, \mathbb{R})$ and

$$\sup_{\substack{x_1, y_1 \in \langle a, b \rangle \\ x_1 \neq y_1}} |[x_1, y_1; f']| \leq 2M.$$

Conversely, let $f : \langle a, b \rangle \rightarrow \mathbb{R}$ be differentiable in $\langle a, b \rangle$, its derivative f' belong to $\text{lip}^1(\langle a, b \rangle, \mathbb{R})$, and let $x, y, z \in \langle a, b \rangle$ be fixed distinct points. By Theorem 1 in [2] (p. 372), a function $(x, y, z) \mapsto [x, y, z; f]$ is symmetric, so we can assume, that $x < y < z$. Then, by the Lagrange Theorem, there exist $\xi_1 \in (x, y)$ and $\xi_2 \in (y, z)$ such that

$$\begin{aligned} \frac{1}{|z - x|} \left| \frac{f(z) - f(y)}{z - y} - \frac{f(y) - f(x)}{y - x} \right| &= \frac{|f'(\xi_1) - f'(\xi_2)|}{|z - x|} \\ &\leq \frac{|f'(\xi_1) - f'(\xi_2)|}{|\xi_1 - \xi_2|} \leq \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f']|. \end{aligned}$$

Thus

$$\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; f]| \leq \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f']|,$$

which ends the proof. $\square$

Remark 1. If $f, g \in \text{lip}^1(\langle a, b \rangle, \mathbb{R})$, then

$$\max\{f, g\}, \min\{f, g\} \in \text{lip}^1(\langle a, b \rangle, \mathbb{R}).$$

Indeed, let $h : \langle a, b \rangle \rightarrow \mathbb{R}$ be given as $h(x) := \max\{f(x), g(x)\}$, $x \in \langle a, b \rangle$. Since $\max\{u, v\} = \frac{u+v+|u-v|}{2}$, we have

$$\begin{aligned} \left| \frac{h(t_2) - h(t_1)}{t_2 - t_1} \right| &\leq \frac{1}{2} \left\{ \left| \frac{f(t_2) - f(t_1)}{t_2 - t_1} \right| + \left| \frac{g(t_2) - g(t_1)}{t_2 - t_1} \right| \right. \\ &\quad \left. + \left| \frac{|f(t_2) - g(t_2)| - |f(t_1) - g(t_1)|}{t_2 - t_1} \right| \right\} \\ &\leq \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f]| + \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; g]| \end{aligned}$$

for all distinct $t_1, t_2 \in \langle a, b \rangle$. Hence

$$\sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; \max\{f, g\}]| \leq \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; f]| + \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; g]|.$$

For $\min\{f, g\}$, the proof and the inequality are the same. $\square$

Remark 2. Let $u, v \in \mathbb{R}$, and $x, z \in \langle a, b \rangle$ $x < z$ be fixed. The function $g : \langle a, b \rangle \rightarrow \mathbb{R}$, given as follows

$$g(t) := \begin{cases} u, & t \in \langle a, x \rangle, \\ \frac{2(v-u)}{(z-x)^2}(t-x)^2 + u, & t \in \left(x, \frac{x+z}{2}\right), \\ \frac{-2(v-u)}{(z-x)^2}(t-z)^2 + v, & t \in \left(\frac{x+z}{2}, z\right), \\ v, & t \in \langle z, b \rangle, \end{cases}$$

belongs to the class $\text{lip}^2(\langle a, b \rangle, \mathbb{R})$ and $\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; g]| \leq 8 \frac{|u-v|}{(z-x)^2}.$

PROOF. If $u = v$, the remark is obvious. Let $u < v$. Then it is easy to check that g is differentiable. Its derivative is equal to

$$g'(t) = \begin{cases} 0, & t \in \langle a, x \rangle, \\ \frac{4(v-u)}{(z-x)^2}(t-x), & t \in (x, \frac{x+z}{2}), \\ \frac{-4(v-u)}{(z-x)^2}(t-z), & t \in (\frac{x+z}{2}, z), \\ 0, & t \in (z, b), \end{cases}$$

and g is of class C^1 . Let us define functions $h_i : \langle a, b \rangle \rightarrow \mathbb{R}$, $i = 1, 2$ by formulas

$$h_1(t) := \frac{4(v-u)}{(z-x)^2}(t-x), \quad h_2(t) := \frac{-4(v-u)}{(z-x)^2}(t-z).$$

Then $g' = \min\{\max\{0, h_1\}, \max\{0, h_2\}\}$, and

$$\sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; g']| \leq \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; h_1]| + \sup_{\substack{t_1, t_2 \in \langle a, b \rangle \\ t_1 \neq t_2}} |[t_1, t_2; h_2]| = 8 \frac{|v-u|}{(z-x)^2}.$$

Thus, by Lemma 1,

$$\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; g]| \leq 8 \frac{|u-v|}{(z-x)^2}.$$

If $v < u$, then define $g_0 : \langle a, b \rangle \rightarrow \mathbb{R}$ as $g_0 = -g$. Then

$$g_0(t) := \begin{cases} -u, & t \in \langle a, x \rangle, \\ \frac{2(u-v)}{(z-x)^2}(t-x)^2 - u, & t \in (x, \frac{x+z}{2}), \\ \frac{-2(u-v)}{(z-x)^2}(t-z)^2 - v, & t \in (\frac{x+z}{2}, z), \\ -v, & t \in (z, b). \end{cases}$$

Since $-u < -v$, $g_0 \in \text{lip}^2(\langle a, b \rangle, \mathbb{R})$ and

$$\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; g_0]| \leq 8 \frac{|-u + v|}{(z - x)^2}. \quad \square$$

A set $C \subseteq X$ is said to be a *convex cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \in \langle 0, \infty \rangle$.

Lemma 2. Let $(X, \|\cdot\|)$ be a real normed space, $C \subseteq X$ be a convex cone and let $x, z \in \langle a, b \rangle$ be fixed, $x < z$. If $u, v \in C$ then the function $\varphi : \langle a, b \rangle \rightarrow X$, given by formula

$$\varphi(t) = \begin{cases} u, & t \in \langle a, x \rangle, \\ \frac{2(v-u)}{(z-x)^2}(t-x)^2 + u, & t \in (x, \frac{x+z}{2}), \\ \frac{-2(v-u)}{(z-x)^2}(t-z)^2 + v, & t \in (\frac{x+z}{2}, z), \\ v, & t \in (z, b), \end{cases}$$

belongs to the class $\text{lip}^2(\langle a, b \rangle, C)$ and $\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; \varphi]\| \leq 8 \frac{\|u-v\|}{(z-x)^2}$.

PROOF. If $t \in \langle x, \frac{x+z}{2} \rangle$ then $0 \leq \left(\frac{t-x}{z-x}\right)^2 \leq \frac{1}{4}$. Thus

$$2 \left(\frac{t-x}{z-x}\right)^2 (v-u) + u = 2 \left(\frac{t-x}{z-x}\right)^2 v + \left(1 - 2 \left(\frac{t-x}{z-x}\right)^2\right) u \in C.$$

Similarly, for $t \in (\frac{x+z}{2}, z)$, $-2 \left(\frac{t-z}{z-x}\right)^2 (v-u) + v \in C$. Hence $\varphi : \langle a, b \rangle \rightarrow C$.

Let us define a function $g : \langle a, b \rangle \rightarrow \mathbb{R}$ by formula

$$g(t) = \begin{cases} 0, & t \in \langle a, x \rangle, \\ \frac{2}{(z-x)^2}(t-x)^2, & t \in (x, \frac{x+z}{2}), \\ \frac{-2}{(z-x)^2}(t-z)^2 + 1, & t \in (\frac{x+z}{2}, z), \\ 1, & t \in (z, b). \end{cases}$$

It is easy to check that $\varphi(t) = (1 - g(t))u + g(t)v$, $t \in \langle a, b \rangle$, and $[t_1, t_2, t_3; \varphi] = [t_1, t_2, t_3; g](v - u)$, for all $t_1, t_2, t_3 \in \langle a, b \rangle$, distinct. By Remark 2, $g \in \text{lip}^2(\langle a, b \rangle, [0, +\infty))$ and $\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; g]| \leq \frac{8}{(z-x)^2}$.

Therefore

$$\|[t_1, t_2, t_3; \varphi]\| \leq 8 \frac{\|u - v\|}{(z-x)^2}. \quad \square$$

Let us consider a real normed space $(Y, \|\cdot\|)$ and a convex cone $S \subset Y$.

Lemma 3. *If a function $f : \langle a, b \rangle \rightarrow S$ belongs to $\text{lip}^2(\langle a, b \rangle, S)$ then f is continuous.*

PROOF. Since $f \in \text{lip}^2(\langle a, b \rangle, S)$, we have

$$\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; f]\| =: M < \infty.$$

Hence

$$\left\| \frac{f(t_3) - f(t_2)}{t_3 - t_2} - \frac{f(t_2) - f(t_1)}{t_2 - t_1} \right\| \leq M |t_3 - t_1|$$

for distinct $t_1, t_2, t_3 \in \langle a, b \rangle$.

Let us fix a point $x \in \langle a, b \rangle$. Choose $y \in \langle a, b \rangle$ such that $x \neq y$.

Then for every $z \in \langle a, b \rangle$, $z \neq x$, $z \neq y$, we have

$$\left\| f(z) - f(y) - \frac{z - y}{y - x} (f(y) - f(x)) \right\| \leq M |z - x| |z - y|.$$

Thus

$$\begin{aligned} \|f(z) - f(x)\| &= \|f(z) - f(y) + f(y) - f(x)\| \\ &= \|f(z) - f(y) + \frac{y - z + (z - x)}{y - x} (f(y) - f(x))\| \\ &\leq \|f(z) - f(y) - \frac{z - y}{y - x} (f(y) - f(x))\| + \left| \frac{z - x}{y - x} \right| \cdot \|f(y) - f(x)\| \\ &\leq M |z - x| |z - y| + \left| \frac{z - x}{y - x} \right| \|f(y) - f(x)\| \end{aligned}$$

for every $z \in \langle a, b \rangle$ such that $z \neq x$ and $z \neq y$. It implies

$$\lim_{z \rightarrow x} \|f(z) - f(x)\| = 0,$$

which proves the continuity of the function f . $\square$

Let X, Z be real vector spaces and C be a convex subset of X . Let $f : C \rightarrow Z$ be an arbitrary function and $h \in X$. The *difference operator* Δ_h with the span h is given by the formula

$$\Delta_h f(x) := f(x + h) - f(x).$$

Thus $\Delta_h f$ is a function mapping the set $\{x \in C : x + h \in C\}$ into the space Z .

The iterates Δ_h^n of Δ_h are given by the recurrence

$$\Delta_h^0 f := f, \quad \Delta_h^{n+1} f := \Delta_h(\Delta_h^n f), \quad n = 0, 1, 2, \dots$$

The expression $\Delta_h^n f$ is a function defined for all $x \in C$ such that $x + nh \in C$. It is easy to see that $x + kh \in C$ for $k = 1, \dots, n-1$ whenever $x, x + nh \in C$.

A function $f : C \rightarrow Z$ is called a *polynomial function* of order at most n if

$$\Delta_h^{n+1} f(x) = 0$$

for every $x \in C$ and $h \in X$ such that $x + (n+1)h \in C$.

Now, we can formulate the main result.

Theorem 1. *Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be real normed spaces, and let $C \subseteq X, S \subseteq Y$ be convex cones and let S be closed. Assume that a function $h : \langle a, b \rangle \times C \rightarrow S$ generates the Nemytskii operator $N : \text{lip}^2(\langle a, b \rangle, C) \rightarrow \text{lip}^2(\langle a, b \rangle, S)$, i.e.,*

$$(3) \quad N\varphi(x) := h(x, \varphi(x)), \quad x \in \langle a, b \rangle, \quad \varphi \in \text{lip}^2(\langle a, b \rangle, C)$$

and that N satisfies the Lipschitz condition

$$(4) \quad \|N\varphi_1 - N\varphi_2\| \leq L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \text{lip}^2(\langle a, b \rangle, C).$$

Then there exist functions $a_o : \langle a, b \rangle \times C \rightarrow S$ and $b_o : \langle a, b \rangle \rightarrow S$ such that

- (i) $a_o(x, \cdot)$ is linear for every $x \in \langle a, b \rangle$,

- (ii) $a_o(x, \cdot)$ is uniformly bounded on C , i.e., there exists a constant $M > 0$ such that $\|a_o(x, y)\| \leq M\|y\|$, $y \in C$ for every $x \in \langle a, b \rangle$,
- (iii) $h(x, y) = a_o(x, y) + b_o(x)$, $x \in \langle a, b \rangle$, $y \in C$,
- (iv) $b_o, a_o(\cdot, y)$ belong to $\text{lip}^2(\langle a, b \rangle, S)$ for all $y \in C$.

Moreover, if $(X, \|\cdot\|)$ is a real Banach space (or the set of the second category) and $\text{int } C \neq \emptyset$ then the function $a_o(x, \cdot) : C \rightarrow S$ is continuous for each $x \in \langle a, b \rangle$.

PROOF. The function $\varphi_0(x) = y_0$, $x \in \langle a, b \rangle$ belongs to $\text{lip}^2(\langle a, b \rangle, C)$ for a fixed $y_0 \in C$. Consequently

$$N\varphi_0 = h(\cdot, y_0) \in \text{lip}^2(\langle a, b \rangle, S) \quad \text{for all } y_0 \in C.$$

Let us fix $x, z \in \langle a, b \rangle$, $x < z$. By (4),

$$\begin{aligned}
 L\|\varphi_1 - \varphi_2\| &\geq \|N\varphi_1 - N\varphi_2\| \\
 &\geq \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; N\varphi_1 - N\varphi_2]\| \geq \left\| \left[x, \frac{x+z}{2}, z; N\varphi_1 - N\varphi_2 \right] \right\| \\
 &= \frac{1}{|z-x|} \left\| \frac{N\varphi_1(z) - N\varphi_2(z) - N\varphi_1\left(\frac{x+z}{2}\right) + N\varphi_2\left(\frac{x+z}{2}\right)}{z - \frac{x+z}{2}} \right. \\
 &\quad \left. - \frac{N\varphi_1\left(\frac{x+z}{2}\right) - N\varphi_2\left(\frac{x+z}{2}\right) - N\varphi_1(x) + N\varphi_2(x)}{\frac{x+z}{2} - x} \right\| \\
 &= \frac{2}{(z-x)^2} \left\| N\varphi_1(z) - N\varphi_2(z) - 2N\varphi_1\left(\frac{x+z}{2}\right) \right. \\
 &\quad \left. + 2N\varphi_2\left(\frac{x+z}{2}\right) + N\varphi_1(x) - N\varphi_2(x) \right\| \\
 &= \frac{2}{(z-x)^2} \left\| h(z, \varphi_1(z)) - h(z, \varphi_2(z)) - 2h\left(\frac{x+z}{2}, \varphi_1\left(\frac{x+z}{2}\right)\right) \right. \\
 &\quad \left. + 2h\left(\frac{x+z}{2}, \varphi_2\left(\frac{x+z}{2}\right)\right) + h(x, \varphi_1(x)) - h(x, \varphi_2(x)) \right\|
 \end{aligned}$$

for every $\varphi_1, \varphi_2 \in \text{lip}^2(\langle a, b \rangle, C)$. Hence

$$\begin{aligned}
 & \left\| h(z, \varphi_1(z)) + 2h\left(\frac{x+z}{2}, \varphi_2\left(\frac{x+z}{2}\right)\right) + h(x, \varphi_1(x)); \right. \\
 (5) \quad & \left. - h(z, \varphi_2(z)) - 2h\left(\frac{x+z}{2}, \varphi_1\left(\frac{x+z}{2}\right)\right) - h(x, \varphi_2(x)) \right\| \\
 & \leq \frac{L}{2}(z-x)^2 \|\varphi_1 - \varphi_2\|
 \end{aligned}$$

for every $\varphi_1, \varphi_2 \in \text{lip}^2(\langle a, b \rangle, C)$.

Let us fix $y_1, y_3, \bar{y}_1, \bar{y}_3 \in C$ and define functions $\varphi_1, \varphi_2 : \langle a, b \rangle \rightarrow X$ as follows

$$\begin{aligned}
 \varphi_1(t) &= \begin{cases} y_1, & t \in \langle a, x \rangle, \\ \frac{2(y_3 - y_1)}{(z-x)^2}(t-x)^2 + y_1, & t \in \left(x, \frac{x+z}{2}\right), \\ \frac{-2(y_3 - y_1)}{(z-x)^2}(t-z)^2 + y_3, & t \in \left(\frac{x+z}{2}, z\right), \\ y_3, & t \in (z, b), \end{cases} \\
 \varphi_2(t) &= \begin{cases} \bar{y}_1, & t \in \langle a, x \rangle, \\ \frac{2(\bar{y}_3 - \bar{y}_1)}{(z-x)^2}(t-x)^2 + \bar{y}_1, & t \in \left(x, \frac{x+z}{2}\right), \\ \frac{-2(\bar{y}_3 - \bar{y}_1)}{(z-x)^2}(t-z)^2 + \bar{y}_3, & t \in \left(\frac{x+z}{2}, z\right), \\ \bar{y}_3, & t \in (z, b). \end{cases}
 \end{aligned}$$

By Lemma 2, $\varphi_1, \varphi_2 \in \text{lip}^2(\langle a, b \rangle, C)$, $\varphi_1(x) = y_1$, $\varphi_1\left(\frac{x+z}{2}\right) = \frac{y_1+y_3}{2}$, $\varphi_1(z) = y_3$, $\varphi_2(x) = \bar{y}_1$, $\varphi_2\left(\frac{x+z}{2}\right) = \frac{\bar{y}_1+\bar{y}_3}{2}$, $\varphi_2(z) = \bar{y}_3$ and

$$\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; \varphi_1 - \varphi_2]\| \leq 8 \frac{\|y_3 - \bar{y}_3 - y_1 + \bar{y}_1\|}{(z-x)^2}.$$

Thus, by (5), we have

$$\begin{aligned}
& \left\| h(z, y_3) - h(z, \bar{y}_3) + 2h\left(\frac{x+z}{2}, \frac{\bar{y}_1 + \bar{y}_3}{2}\right) \right. \\
& \quad \left. - 2h\left(\frac{x+z}{2}, \frac{y_1 + y_3}{2}\right) + h(x, y_1) - h(x, \bar{y}_1) \right\| \\
& \leq \frac{L}{2}(z-x)^2 \left(\|y_1 - \bar{y}_1\| + \|y_3 - \bar{y}_3\| + 8 \frac{\|y_3 - \bar{y}_3 - y_1 + \bar{y}_1\|}{(z-x)^2} \right) \\
& = 4L\|y_3 - \bar{y}_3 - y_1 + \bar{y}_1\| + \frac{L}{2}(z-x)^2 (\|y_3 - \bar{y}_3\| + \|y_1 - \bar{y}_1\|)
\end{aligned}$$

for all $x, z \in \langle a, b \rangle$, $x < z$, $y_1, y_3, \bar{y}_1, \bar{y}_3 \in C$.

By Lemma 3, the function $h(\cdot, y_0)$ is continuous in $\langle a, b \rangle$ for every $y_0 \in C$. Letting $z \rightarrow x$, we get the inequality:

$$\begin{aligned}
(6) \quad & \left\| h(x, y_3) - h(x, \bar{y}_3) + 2h\left(x, \frac{\bar{y}_1 + \bar{y}_3}{2}\right) - 2h\left(x, \frac{y_1 + y_3}{2}\right) \right. \\
& \quad \left. + h(x, y_1) - h(x, \bar{y}_1) \right\| \leq 4L\|y_3 - \bar{y}_3 - y_1 + \bar{y}_1\|
\end{aligned}$$

for all $x \in \langle a, b \rangle$, $y_1, y_3, \bar{y}_1, \bar{y}_3 \in C$.

Now, let us fix $y \in C$, $h \in X$ such that $y + 3h \in C$. Putting in (6), $y_3 := y + 3h$, $\bar{y}_3 := y + 2h$, $y_1 := y + h$, $\bar{y}_1 := y$ we get

$$\|h(x, y + 3h) + 3h(x, y + h) - 3h(x, y + 2h) - h(x, y)\| = 0$$

for $x \in \langle a, b \rangle$, $y \in C$, $h \in X$ such that $y + 3h \in C$.

This means that for every $x \in \langle a, b \rangle$ the function $h(x, \cdot) : C \rightarrow S$ is a polynomial function of order at most 2. According to Corollary 2 in [1], there exist functions $\psi_0 : \langle a, b \rangle \rightarrow Y$ and $\psi_k : \langle a, b \rangle \times C^k \rightarrow Y$, $k = 1, 2$ such that $\psi_1(x, \cdot)$ is additive and $\psi_2(x, \cdot, \cdot)$ is biadditive, $x \in \langle a, b \rangle$ and

$$h(x, y) = \psi_0(x) + \psi_1(x, y) + \psi_2(x, y, y)$$

for all $x \in \langle a, b \rangle$, $y \in C$.

Let us fix $y \in C$ and put in (6) $y_3 := y$, $y_1 = \bar{y}_1 = \bar{y}_3 := 0$. Then

$$\begin{aligned} & \left\| h(x, y) - h(x, 0) + 2h(x, 0) - 2h\left(x, \frac{y}{2}\right) + h(x, 0) - h(x, 0) \right\| \\ &= \left\| h(x, y) - 2h\left(x, \frac{y}{2}\right) + h(x, 0) \right\| \leq 4L\|y\| \end{aligned}$$

for all $x \in \langle a, b \rangle$, $y \in C$.

Since $\psi_2(x, \cdot, \cdot)$ is biadditive and $\|\psi_2(x, y, y)\| \leq 8L\|y\|$, $x \in \langle a, b \rangle$, $y \in C$, we conclude that $\psi_2(x, y, y) = 0$, $x \in \langle a, b \rangle$, $y \in C$. Hence

$$(7) \quad h(x, \cdot) = \psi_0(x) + \psi_1(x, \cdot), \quad x \in \langle a, b \rangle,$$

is polynomial of order at most one, for each $x \in \langle a, b \rangle$. There exist unique constants $b_o(x) \in Y$ and the additive functions $a_o(x, \cdot) : C \rightarrow Y$ such that

$$(8) \quad h(x, y) = a_o(x, y) + b_o(x) \quad \text{for } x \in \langle a, b \rangle, y \in C.$$

It is easy to check, that $b_o(x) \in S$ and $a_o(x, \cdot) : C \rightarrow S$.

To prove the linearity of the function $a_o(x, \cdot)$, $x \in \langle a, b \rangle$, let us fix $x, z \in \langle a, b \rangle$, $x \neq z$, $y_1, y_3 \in C$ such that $y_1 - y_3 \in C$. Consider a function $\bar{\varphi}_1 : \langle a, b \rangle \rightarrow C$ given as follows

$$\bar{\varphi}_1(t) := \frac{y_1 - y_3}{(z - x)^2}(t - x)^2 + y_3, \quad t \in \langle a, b \rangle.$$

Then $\bar{\varphi}_1(x) = y_3$, $\bar{\varphi}_1(z) = y_1$, $\bar{\varphi}_1(\frac{x+z}{2}) = \frac{y_1 + 3y_3}{4}$, and

$$\begin{aligned} & \|[t_1, t_2, t_3; \bar{\varphi}_1]\| \\ &= \frac{\|y_3 - y_1\|}{(z - x)^2} \frac{1}{|t_3 - t_1|} \left| \frac{(t_3 - x)^2 - (t_2 - x)^2}{t_3 - t_2} - \frac{(t_2 - x)^2 - (t_1 - x)^2}{t_2 - t_1} \right| \\ &= \frac{\|y_3 - y_1\|}{(z - x)^2}, \quad \text{for distinct } t_1, t_2, t_3 \in \langle a, b \rangle. \end{aligned}$$

Hence $\bar{\varphi}_1 \in \text{lip}^2(\langle a, b \rangle, C)$. Let $\bar{\varphi}_2(t) = 0$, $t \in \langle a, b \rangle$. By (5), we have

$$\begin{aligned} & \left\| h(z, y_1) - h(x, 0) + 2h\left(\frac{x+z}{2}, 0\right) \right. \\ & \quad \left. + h(x, y_3) - 2h\left(\frac{x+z}{2}, \frac{y_1 + 3y_3}{4}\right) - h(x, 0) \right\| \leq \frac{L}{2}(z - x)^2 \|\bar{\varphi}_1\| \\ & \leq \frac{L}{2} \|y_3 - y_1\| (1 + (a - x)^2 + (b - x)^2) + L\|y_3\|(z - x)^2 \end{aligned}$$

for all $x, z \in \langle a, b \rangle$, $x \neq z$, $y_1, y_3 \in C$, such that $y_1 - y_3 \in C$. Letting $z \rightarrow x$, by the continuity of $h(\cdot, y)$, $y \in C$ we obtain

$$\begin{aligned} & \left\| h(x, y_1) - h(x, 0) + 2h(x, 0) - 2h\left(x, \frac{y_1 + 3y_3}{4}\right) + h(x, y_3) - h(x, 0) \right\| \\ & \leq \frac{L}{2} (1 + (a - x)^2 + (b - x)^2) \|y_3 - y_1\| \leq \frac{L}{2} (1 + (a - b)^2) \|y_3 - y_1\| \end{aligned}$$

for $x \in \langle a, b \rangle$, $y_1, y_3 \in C$, $y_1 - y_3 \in C$. Thus, by (8)

$$(9) \quad \|a_o(x, y_1) - a_o(x, y_3)\| \leq L (1 + (a - b)^2) \|y_3 - y_1\|$$

for all $x \in \langle a, b \rangle$, $y_1, y_3 \in C$ such that $y_1 - y_3 \in C$. In particular

$$\|a_o(x, y)\| \leq L (1 + (a - b)^2) \|y\|$$

for $x \in \langle a, b \rangle$, $y \in C$, which proves (ii).

To prove (i), let us fix $\lambda > 0$. Then $\lambda = \lim_{n \rightarrow \infty} \lambda_n$, $\lambda_n \in \mathbb{Q} \cap \langle 0, \infty \rangle$, $\lambda_n \geq \lambda$, $n \in \mathbb{N}$. Therefore $\lambda_n y - \lambda y = (\lambda_n - \lambda)y \in C$ for $y \in C$, $n \in \mathbb{N}$, and by (9)

$$\begin{aligned} & \|\lambda_n a_o(x, y) - a_o(x, \lambda y)\| \\ & = \|a_o(x, \lambda_n y) - a_o(x, \lambda y)\| \leq L (1 + (a - b)^2) |\lambda_n - \lambda| \|y\| \end{aligned}$$

for $n \in \mathbb{N}$, $y \in C$, $x \in \langle a, b \rangle$. Letting $n \rightarrow \infty$, we get $a_o(x, \lambda y) = \lambda a_o(x, y)$ for all $y \in C$, $x \in \langle a, b \rangle$, $\lambda > 0$.

Since $\tilde{\varphi} = 0$ belongs to $\text{lip}^2(\langle a, b \rangle, C)$,

$$b_o(\cdot) = a_o(\cdot, \tilde{\varphi}) + b_o(\cdot) = h(\cdot, \tilde{\varphi}) = N\tilde{\varphi}(\cdot) \in \text{lip}^2(\langle a, b \rangle, S).$$

Similarly, let $\tilde{\varphi}_1 = 0$, $\tilde{\varphi}_2 = y$, where $y \in C$ is fixed. Then, by the Lipschitz condition

$$\begin{aligned} \|[t_1, t_2, t_3; a_o(\cdot, y)]\| &= \|[t_1, t_2, t_3; a_o(\cdot, \tilde{\varphi}_2) - a_o(\cdot, \tilde{\varphi}_1)]\| \\ &= \|[t_1, t_2, t_3; N\tilde{\varphi}_2 - N\tilde{\varphi}_1]\| \leq L \|\tilde{\varphi}_1 - \tilde{\varphi}_2\| = 2L\|y\| \end{aligned}$$

for all $t_1, t_2, t_3 \in \langle a, b \rangle$, which imply $a_o(\cdot, y) \in \text{lip}^2(\langle a, b \rangle, S)$ and ends the first part of the proof.

Now, let us assume that $(X, \|\cdot\|)$ is a real Banach space, and $\text{int } C \neq \emptyset$. There exists an open ball $B \subset C$. Fix $x \in \langle a, b \rangle$. Then the function $f : C \rightarrow Y$ given by

$$f(y) := a_o(x, y), \quad y \in C$$

is linear. By Theorem 2 ([1]), there exists the unique linear extension $\bar{f} : X \rightarrow Y$ of the function f . The function $\bar{f}$ is linear and bounded on B , which is a set of the second category. Therefore, by Theorem 5 in [6], the function $\bar{f}$ is continuous, so the function f does, which ends the proof. $\square$

Now we formulate a multivalued version of Theorem 1.

Let $(Y, \|\cdot\|)$ be a real normed space. We denote as $n(Y)$ the family of all nonempty subsets of Y , as $cc(Y)$ the family of all convex and compact members of $n(Y)$, and as θ the singleton $\{0\}$. The class $cc(Y)$ may be considered as a metric space with the Hausdorff distance δ given as follows

$$\delta(A, B) := \inf \{t > 0 : A \subseteq B + tS, B \subseteq A + tS\},$$

where S is closed unit ball in Y . Of course, $\delta(A, \theta) = \sup\{\|a\| : a \in A\}$ for $A \in cc(Y)$. We define

$$\|A\| := \delta(A, \theta).$$

Let $F : \langle a, b \rangle \rightarrow cc(Y)$ be a set-valued function (abbreviated to s.v. function in the sequel). Hans Rådström defined the equivalence relation $\sim$ on $cc(Y)$ (see [8]) by stating $(A, B) \sim (C, D)$ if $A + D = B + C$. The equivalence class containing (A, B) is denoted by $[A, B]_\sim$. The quotient space $\mathcal{Y} = (cc(Y))^2 / \sim$, with addition defined by

$$[A, B]_\sim + [D, E]_\sim := [A + D, B + E]_\sim,$$

and scalar multiplication

$$\lambda[A, B]_\sim := \begin{cases} [\lambda A, \lambda B]_\sim, & \lambda \geq 0, \\ [(-\lambda)B, (-\lambda)A]_\sim, & \lambda < 0, \end{cases}$$

is a real normed space with the norm defined by the Hausdorff metric

$$\|[A, B]_\sim\| := \delta(A, B), \quad A, B \in cc(Y).$$

Let define $\mathcal{S} := \{[A, \theta]_\sim, A \in cc(Y)\}$. It is obvious that $\mathcal{S}$ is a convex cone in the space $\mathcal{Y}$. By Lemma 6 in [12], the cone $\mathcal{S}$ is closed in the normed space $\mathcal{Y}$. Consider

$$\text{Lip}^2(\langle a, b \rangle, cc(Y)) := \text{lip}^2(\langle a, b \rangle, \mathcal{S}).$$

Then

$$\begin{aligned} & \text{Lip}^2(\langle a, b \rangle, cc(Y)) \\ &= \left\{ F : \langle a, b \rangle \rightarrow cc(Y) : \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; [F, \theta]_\sim]\| < \infty \right\}, \end{aligned}$$

where $[F, \theta]_\sim$ denotes the function $\langle a, b \rangle \ni x \mapsto [F(x), \theta]_\sim \in \mathcal{S}$ and $(\text{Lip}^2(\langle a, b \rangle, cc(Y)), d)$ is a metric space with the metric given

$$d(F, G) := \delta(F(a), G(a)) + \delta(F(b), G(b)) + \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} \|[t_1, t_2, t_3; [F, G]_\sim]\|.$$

By the definition, $F \in \text{Lip}^2(\langle a, b \rangle, cc(Y))$ iff the function $\langle a, b \rangle \ni x \mapsto [F(x), \theta]_\sim \in \mathcal{S}$ belongs to the class $\text{lip}^2(\langle a, b \rangle, \mathcal{S})$.

Consider an s.v. function

$$H : \langle a, b \rangle \times C \rightarrow cc(Y)$$

which generates the Nemytskii operator

$$\mathbf{N} : \text{lip}^2(\langle a, b \rangle, C) \rightarrow \text{Lip}^2(\langle a, b \rangle, cc(Y)),$$

$$\mathbf{N}\varphi(x) := H(x, \varphi(x)), \quad x \in \langle a, b \rangle, \quad \varphi \in \text{lip}^2(\langle a, b \rangle, C).$$

Assume that $\mathbf{N}$ satisfies the Lipschitz condition

$$d(\mathbf{N}\varphi_1, \mathbf{N}\varphi_2) \leq L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \text{lip}^2(\langle a, b \rangle, C).$$

Then the function

$$h(x, y) : \langle a, b \rangle \times C \rightarrow \mathcal{S}$$

given by

$$h(x, y) := [H(x, y), \theta]_\sim, \quad x \in \langle a, b \rangle, \quad y \in C$$

generates the Nemytskii operator

$$N : \text{lip}^2(\langle a, b \rangle, C) \rightarrow \text{lip}^2(\langle a, b \rangle, \mathcal{S}),$$

$$N\varphi(x) = h(x, \varphi(x)) = [H(x, \varphi(x)), \theta]_{\sim} = [N\varphi(x), \theta]_{\sim}.$$

The operator N is Lipschitzian, so by Theorem 1, there exist functions $a_o : \langle a, b \rangle \times C \rightarrow \mathcal{S}$ and $b_o : \langle a, b \rangle \rightarrow \mathcal{S}$ such that

- (i) $a_o(x, \cdot)$ is linear for all $x \in \langle a, b \rangle$,
- (ii) $a_o(x, \cdot)$ is uniformly bounded on the cone C , $x \in \langle a, b \rangle$,
- (iii) $h(x, y) = a_o(x, y) + b_o(x)$, $x \in \langle a, b \rangle$, $y \in C$,
- (iv) $b_o, a_o(\cdot, y) \in \text{lip}^2(\langle a, b \rangle, \mathcal{S})$ for all $y \in C$.

By the definition of $\mathcal{S}$, there exist s.v. functions

$$A : \langle a, b \rangle \times C \rightarrow cc(Y),$$

$$B : \langle a, b \rangle \rightarrow cc(Y)$$

such that

$$a_o(x, y) = [A(x, y), \theta]_{\sim}, \quad b_o(x) = [B(x), \theta]_{\sim}, \quad x \in \langle a, b \rangle, \quad y \in C.$$

The linearity of the function $A(x, \cdot)$ follows from the linearity of $a_o(x, \cdot)$. By (ii) there exists a constant $M > 0$ such that $\|a_o(x, y)\| \leq M\|y\|$, $x \in \langle a, b \rangle$. Thus, for all $x \in \langle a, b \rangle$,

$$\|A(x, y)\| = \sup\{\|z\|; z \in A(x, y)\} = \|[A(x, y), \theta]_{\sim}\| = \|a_o(x, y)\| \leq M\|y\|.$$

By (iii), (iv) and the definition of $\mathcal{S}$, $H(x, y) = A(x, y) + B(x)$, $x \in \langle a, b \rangle$, $y \in C$ and $A(\cdot, y), B \in \text{Lip}^2(\langle a, b \rangle, cc(Y))$.

If the values of an s.v. function F are compact, then the continuity of F (cf. [7]) is equivalent to the continuity of the function F treated as an ordinary function with values in the metric space $c(Y)$ -nonempty and compact subsets of the space Y with the Hausdorff metric. So, if we assume that $(X, \|\cdot\|)$ is a Banach space and $\text{int } C \neq \emptyset$, then continuity of the function $a_o(x, \cdot)$ implies the continuity of the function $A(x, \cdot)$, for each $x \in \langle a, b \rangle$.

So, we have the following collorary:

Corollary 1. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be real normed spaces, and let $C \subseteq X$ be a convex cone. Let a function $H : \langle a, b \rangle \times C \rightarrow cc(Y)$ generates the Nemytskii operator $N : \text{lip}^2(\langle a, b \rangle, C) \rightarrow \text{Lip}^2(\langle a, b \rangle, cc(Y))$, i.e.,*

$$N\varphi(x) := H(x, \varphi(x)), \quad x \in \langle a, b \rangle, \quad \varphi \in \text{lip}^2(\langle a, b \rangle, C)$$

and let N satisfies the Lipschitz condition

$$d(N\varphi_1, N\varphi_2) \leq L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \text{lip}^2(\langle a, b \rangle, C).$$

Then there exist s.v. functions $A : \langle a, b \rangle \times C \rightarrow cc(Y)$ and $B : \langle a, b \rangle \rightarrow cc(Y)$ such that

- (i) $A(x, \cdot)$ is linear for every $x \in \langle a, b \rangle$,
- (ii) $A(x, \cdot)$ is uniformly bounded on C , i.e., there exists a constant $M > 0$ such that $\|A(x, y)\| \leq M\|y\|$, $y \in C$ for every $x \in \langle a, b \rangle$,
- (iii) $H(x, y) = A(x, y) + B(x)$, $x \in \langle a, b \rangle$, $y \in C$,
- (iv) $B, A(\cdot, y)$ belong to $\text{Lip}^2(\langle a, b \rangle, cc(Y))$ for all $y \in C$.

Moreover, if $(X, \|\cdot\|)$ is a real Banach space (or the set of the second category) and $\text{int } C \neq \emptyset$ then the function $A(x, \cdot) : C \rightarrow cc(Y)$ is continuous for each $x \in \langle a, b \rangle$.

In the real case, we have

Theorem 2. *Let $C = \mathbb{R}$ or $C = [0, \infty)$ or $C = (-\infty, 0]$ and let $(Y, \|\cdot\|)$ be a real normed space. A function $H : \langle a, b \rangle \times C \rightarrow cc(Y)$ generates the Nemytskii operator $N : \text{lip}^2(\langle a, b \rangle, C) \rightarrow \text{Lip}^2(\langle a, b \rangle, cc(Y))$, satisfying the Lipschitz condition (4), if and only if there exist s.v. functions $A, B : \langle a, b \rangle \rightarrow cc(Y)$ such that*

- (i) $A, B \in \text{Lip}^2(\langle a, b \rangle, cc(Y))$,
- (ii) A is bounded, i.e., there exists a constant $M > 0$ such that $\|A(x)\| \leq M$ for $x \in \langle a, b \rangle$,
- (iii) $H(x, y) = A(x)y + B(x)$; $x \in \langle a, b \rangle$, $y \in C$.

PROOF. Assume that H generates the Nemytskii operator, which is Lipschitzian. Let us fix $x \in \langle a, b \rangle$. By the last theorem, $A(x, \cdot) : C \rightarrow cc(Y)$ is a linear function. Hence, there exists a unique set $A(x) \in cc(Y)$ such that

$$A(x, y) = A(x)y \quad \text{for all } y \in C.$$

Boundedness of the function

$$\langle a, b \rangle \ni x \mapsto A(x) \in cc(Y)$$

follows from (ii) of Corollary 1. Similary, (i) and (iii) are clear from Corollary 1.

Conversely, let us fix $\varphi \in \text{lip}^2(\langle a, b \rangle, C)$. Then the map

$$\langle a, b \rangle \ni x \mapsto N\varphi(x) = \varphi(x)A(x) + B(x) \in cc(Y),$$

belongs to $\text{Lip}^2(\langle a, b \rangle, cc(Y))$.

Indeed, for fixed distinct points $t_1, t_2, t_3 \in \langle a, b \rangle$, we have

$$\begin{aligned} & \| [t_1, t_2, t_3; [N\varphi, \theta]_\sim] \| \\ & \leq \| [t_1, t_2, t_3; [B, \theta]_\sim] \| + \| [t_1, t_2, t_3; [\varphi A, \theta]_\sim] \|. \end{aligned}$$

Let us consider the case $C = (-\infty, 0]$. Then

$$\begin{aligned} & \| [t_1, t_2, t_3; [\varphi A, \theta]_\sim] \| \\ & \leq \frac{1}{|t_3 - t_1|} \left\| \frac{[A(t_2)\varphi(t_3), A(t_2)\varphi(t_2)]_\sim}{t_3 - t_2} - \frac{[A(t_2)\varphi(t_2), A(t_2)\varphi(t_1)]_\sim}{t_2 - t_1} \right\| \\ & \quad + \frac{1}{|t_3 - t_1|} \left\| \frac{[A(t_3)\varphi(t_3), A(t_2)\varphi(t_3)]_\sim}{t_3 - t_2} - \frac{[A(t_2)\varphi(t_3), A(t_1)\varphi(t_3)]_\sim}{t_2 - t_1} \right\| \\ & \quad + \frac{1}{|t_3 - t_1|} \left\| \frac{[A(t_2)\varphi(t_3), A(t_1)\varphi(t_3)]_\sim}{t_2 - t_1} - \frac{[A(t_2)\varphi(t_1), A(t_1)\varphi(t_1)]_\sim}{t_2 - t_1} \right\| \\ & = |[t_1, t_2, t_3; \varphi] \cdot \| [-A(t_2), \theta]_\sim \| \\ & \quad + |\varphi(t_3)| \cdot \frac{1}{|t_3 - t_1|} \left\| \frac{[-A(t_3), -A(t_2)]_\sim}{t_3 - t_2} - \frac{[-A(t_2), -A(t_1)]_\sim}{t_2 - t_1} \right\| \\ & \quad + \left| \frac{\varphi(t_3) - \varphi(t_1)}{t_3 - t_1} \right| \cdot \left\| \frac{[-A(t_2), -A(t_1)]_\sim}{t_2 - t_1} \right\| \end{aligned}$$

for $t_1, t_2, t_3 \in \langle a, b \rangle$, distinct. Since,

$$\begin{aligned} |\varphi(t_3)| & \leq \left| \frac{\varphi(t_3) - \varphi(b)}{t_3 - b} \right| (b - a) + |\varphi(b)| \\ & \leq \left| \frac{\varphi(t_3) - \varphi(b)}{t_3 - b} - \frac{\varphi(b) - \varphi(a)}{b - a} \right| (b - a) + \left| \frac{\varphi(b) - \varphi(a)}{b - a} \right| (b - a) + |\varphi(b)| \end{aligned}$$

$$\leq |[a, b, t_3; \varphi]|(b-a)^2 + |\varphi(a)| + 2|\varphi(b)| \leq ((b-a)^2 + 2) \|\varphi\|,$$

$$\begin{aligned} & \left\| \frac{[-A(t_2), -A(t_1)]_{\sim}}{t_2 - t_1} \right\| \\ & \leq \left\| \frac{[-A(t_2), -A(t_1)]_{\sim}}{t_2 - t_1} - \frac{[-A(t_1), -A(a)]_{\sim}}{t_1 - a} \right\| + \left\| \frac{[-A(t_1), -A(a)]_{\sim}}{t_1 - a} \right\| \\ & \leq \|[a, t_1, t_2; [-A, \theta]_{\sim}]\|(b-a) + \|[b, a, t_1; [-A, \theta]_{\sim}]\|(b-a) \\ & \quad + \frac{\delta(A(a), A(b))}{b-a} \leq \left(2(b-a) + \frac{1}{b-a}\right) \|[-A, \theta]_{\sim}\|, \end{aligned}$$

and similarly,

$$\begin{aligned} & \|[-A(t_2), \theta]_{\sim}\| \leq ((b-a)^2 + 2) \|[-A, \theta]_{\sim}\|, \\ & \left| \frac{\varphi(t_3) - \varphi(t_1)}{t_3 - t_1} \right| \leq \left(2(b-a) + \frac{1}{b-a}\right) \|\varphi\|, \end{aligned}$$

the following inequality holds:

$$\begin{aligned} & \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, \ j \neq i}} \|[t_1, t_2, t_3; [\varphi A, \theta]_{\sim}]\| \\ & \leq 2(2 + (b-a)^2) \|[-A, \theta]_{\sim}\| \|\varphi\| + \left(2(b-a) + \frac{1}{b-a}\right)^2 \|[-A, \theta]_{\sim}\| \|\varphi\| \\ & = M_A \|\varphi\| \end{aligned}$$

for all $\varphi \in \text{lip}^2(\langle a, b \rangle, C)$, where M_A is a constant independent on φ . The last inequality shows that $N\varphi \in \text{Lip}^2(\langle a, b \rangle, cc(Y))$ as well as N is Lipschitzian. The same proof works for $C = [0, \infty)$. For $C = \mathbb{R}$, the s.v. function A is an ordinary function $A : \langle a, b \rangle \rightarrow Y$. Then $[A(u), A(v)]_{\sim} = [A(u) - A(v), 0]_{\sim}$, which simplifies the above considerations. $\square$

Since every function belonging to the class $\text{lip}^2(\langle a, b \rangle, C)$ is bounded, the following consequence of Theorem 2 holds:

Corollary 2. *Assume that $C = \mathbb{R}$ or $C = [0, \infty)$ or $C = (-\infty, 0]$ and $(Y, \|\cdot\|)$ is a real normed space. A function $h : \langle a, b \rangle \times C \rightarrow Y$ generates*

the Nemytskii operator $N : \text{lip}^2(\langle a, b \rangle, C) \rightarrow \text{lip}^2(\langle a, b \rangle, Y)$, satisfying the Lipschitz condition

$$(10) \quad \|N\varphi_1 - N\varphi_2\| \leq L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \text{lip}^2(\langle a, b \rangle, C)$$

if and only if there exist functions $a_o, b_o : \langle a, b \rangle \rightarrow Y$, $a_o, b_o \in \text{lip}^2(\langle a, b \rangle, Y)$ such that

$$h(x, y) = a_o(x)y + b_o(x) \quad \text{for } x \in \langle a, b \rangle, y \in C.$$

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