

On a class of QR-submanifolds of quaternion space forms

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Abstract. Let M be a QR-submanifolds (cf. A. BEJANCU [2]) of a quaternion space form $N(c)$ such that every geodesic of M is a circle of $N(c)$. Then we prove that either $c = 0$ or M is a totally real submanifold.

1. Introduction

Let N be a quaternion Kaehlerian manifold of real dimension $4n$. Then there exist on N a Riemannian metric g and a vector bundle V of tensors of type (1,1) with a local basis of Hermitian structures J_1, J_2, J_3 such that $J_1 J_2 = -J_2 J_1 = J_3$. Moreover, for each local section S of V and vector field X on N , $\tilde{\nabla}_X S$ is also a section of V , where $\tilde{\nabla}$ is the Levi-Civita connection on N with respect to g . If the quaternion sectional curvature of N is a constant c then we say that N is a quaternion space form and denote it by $N(c)$. As it is well known (cf. K. YANO and M. KON [10], p. 172) the curvature tensor \tilde{R} of $\tilde{\nabla}$ on $N(c)$ is given by

$$(1.1) \quad \begin{aligned} \tilde{R}(X, Y)Z &= \frac{c}{4} \left\{ g(Y, Z)X - g(X, Z)Y \right. \\ &\left. + \sum_{a=1}^3 \{ g(J_a Y, Z)J_a X - g(J_a X, Z)J_a Y + 2g(X, J_a Y)J_a Z \} \right\} \end{aligned}$$

for any $X, Y, Z \in \Gamma(TN(c))$.

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Throughout the paper we use the following notations:

$F(N)$ – the algebra of smooth functions on N ,

$\Gamma(E)$ – the $F(N)$ -module of smooth sections of a vector bundle E over N . The same notations will be used for any other manifold.

Next, we consider an m -dimensional Riemannian manifold M isometrically immersed in N . Then M is said to be a *quaternion-real submanifold* (*QR-submanifold*) (cf. A. BEJANCU [2]) if there exist a vector subbundle ν of the normal bundle TM^\perp such that

$$J_a(\nu_x) = \nu_x \quad \text{and} \quad J_a(\nu_x^\perp) \subset T_x M,$$

for any $x \in M$ and $a \in \{1, 2, 3\}$, where ν^\perp is the complementary orthogonal vector bundle to ν in TM^\perp . If, in particular, $\nu = TM^\perp$ (resp. $\nu = \{0\}$), M is said to be a *quaternion submanifold* (cf. B.Y. CHEN [4]) (resp. *anti-quaternion submanifold*, cf. J.S. PAK [8]).

Denote by s the rank of ν^\perp , that is, each fibre of ν^\perp is of dimension s . Then consider the s -dimensional vector subspaces $D_{a,x} = J_a(\nu_x^\perp)$, $a \in \{1, 2, 3\}$ of $T_x M$ which are mutually orthogonal. Thus we obtain a globally defined distribution

$$D^\perp : x \rightarrow D_x^\perp = D_{1,x} \oplus D_{2,x} \oplus D_{3,x},$$

on M . Also, we have

$$J_a(D_{a,x}) = \nu_x^\perp \quad \text{and} \quad J_a(D_{b,x}) = D_{c,x},$$

for any $x \in M$, where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Denote by D the complementary orthogonal distribution to D^\perp in TM . It is easy to see that D is a quaternion distribution on M , that is, $J_a(D_x) = D_x$ for any $x \in M$ and $a \in \{1, 2, 3\}$.

Finally, we recall from K. NOMIZU and K. YANO [6] the notion of circle in N . A curve $C : x = x(t)$ with arc-length parameter t in N is said to be a *circle* if there exists a field of unit vectors Y_t along C , which, together with the unit tangent vectors X_t , satisfies the differential equations

$$(1.2) \quad \tilde{\nabla}_t X_t = kY_t \quad \text{and} \quad \tilde{\nabla}_t Y_t = -kX_t,$$

where k is a positive constant.

The main purpose of this note is to prove the following

Theorem 1. *Let M be a QR-submanifold of $N(c)$, such that each geodesic of M is a circle in $N(c)$. Then either $c = 0$ or $D = \{0\}$.*

2. Preliminaries

First, from the general theory of submanifolds we recall the formulas of Gauss and Weingarten (cf. B.Y. CHEN [3], p. 39):

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM)$$

and

$$(2.2) \quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, \quad \forall X \in \Gamma(TM), \xi \in \Gamma(TM^\perp),$$

respectively, where ∇ , ∇^\perp , h and A_ξ are the Levi-Civita connection on M , the normal connection of M , the second fundamental form and the shape operator of M , respectively. Also, h and A_ξ are related by

$$(2.3) \quad g(h(X, Y), \xi) = g(A_\xi X, Y), \quad \forall X, Y \in \Gamma(TM), \xi \in \Gamma(TM^\perp).$$

The covariant derivative of h is defined by

$$(2.4) \quad (\nabla_X h)(Y, Z) = \nabla_X^\perp (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \\ \forall X, Y, Z \in \Gamma(TM).$$

Then the Codazzi equation is given by

$$(2.5) \quad \tilde{R}(X, Y)Z^\perp = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \quad \forall X, Y, Z \in \Gamma(TM),$$

where $\{*\}^\perp$ means the normal part of $*$.

Now, suppose M is a QR-submanifold of a quaternion Kaehlerian manifold N . Then according to C.L. BEJAN [1] the second fundamental form of M satisfies

$$(2.6) \quad g(h(X, Y), \xi) = 0, \quad \forall X, Y \in \Gamma(D), \xi \in \Gamma(v).$$

Let $C : x = x(t)$ be a geodesic of M , that is, $\nabla_t X_t = 0$. Thus from (2.1) we deduce

$$\tilde{\nabla}_t X_t = h(X_t, X_t),$$

which, together with (2.2), implies

$$(2.7) \quad \tilde{\nabla}_t^2 X_t = -A_{h(X_t, X_t)} X_t + \nabla_t^\perp (h(X_t, X_t)).$$

Suppose now C is a circle in N . Then by using (2.1) and (1.2) we deduce

$$(2.8) \quad \tilde{\nabla}_t^2 X_t = -k^2 X_t.$$

Comparing (2.7) and (2.8) we infer

$$(2.9) \quad A_{h(X_t, X_t)} X_t = k^2 X_t,$$

and

$$(2.10) \quad \nabla_t^\perp (h(X_t, X_t)) = 0.$$

Consider $t = 0$. Using (2.3) and (2.9) we obtain

$$g(h(X, X), h(X, X)) = k^2$$

for any unit tangent vector X at a point $x \in M$. Thus there exists a non-zero function λ such that

$$(2.11) \quad g(h(X, X), h(X, X)) = \lambda^2,$$

for any unit vector field X on M . According to O'NEILL [7], the immersion of M in $N(c)$ is λ -isotropic. From (2.11) we deduce

$$(2.12) \quad g(h(X, X), h(X, X)) = \lambda^2 g(X, X)g(X, X), \quad \forall X \in \Gamma(TM),$$

which is equivalent to

$$(2.13) \quad \begin{aligned} &g(h(X, Y), h(Z, U)) + g(X, Z), h(U, Y)) + g(X, U), h(Y, Z)) \\ &= \lambda^2 \{g(X, Y)g(Z, U) + g(X, Z)g(U, Y) + g(X, U)g(Y, Z)\}, \end{aligned}$$

for any $X, Y, Z, U \in \Gamma(TM)$. Also, from (2.10) and (2.4), taking into account that C is a geodesic, we deduce:

$$(2.14) \quad (\nabla_X h)(X, X) = 0, \quad \forall X \in \Gamma(TM),$$

which is equivalent to

$$(2.15) \quad \begin{aligned} &(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) + (\nabla_Z h)(X, Y) = 0, \\ &\forall X, Y, Z \in \Gamma(TM). \end{aligned}$$

Remark 1. K. NOMIZU [5] obtained both (2.11) and (2.14) for a Kaehler submanifold of a complex projective space. Actually, it is easy to see that the above calculations hold good for a Riemannian submanifold whose geodesics are circles in the ambient space.

3. Proof of the Theorem

First, differentiating (2.11) and taking account of (2.14) we deduce $X(\lambda) = 0$ for any $X \in \Gamma(TM)$ that is, λ is locally constant on M . Then we take $Y = Z \in \Gamma(D)$ and $X \in \Gamma(D)$ in (2.15) and obtain

$$(\nabla_X h)(Y, Y) + 2(\nabla_Y h)(X, Y) = 0.$$

On the other hand, by using (1.1) in (2.5) we derive

$$(\nabla_X h)(Y, Y) - (\nabla_Y h)(X, Y) = 0, \quad \forall X, Y \in \Gamma(D).$$

Hence we have

$$(3.1) \quad (\nabla_X h)(Y, Y) = 0, \quad \forall X, Y \in \Gamma(D).$$

Next, we take $Y = Z = U \in \Gamma(D)$, $X = J_1\eta$ for $\eta \in \Gamma(\nu^\perp)$ in (2.13) and differentiate (2.13) with respect to $W \in \Gamma(D)$. Then taking into account that λ is locally constant and by using (3.1) we obtain

$$(3.2) \quad g((\nabla_W h)(J_1\eta, Y), h(Y, Y)) = 0, \quad \forall Y, W \in \Gamma(D), \eta \in \Gamma(\nu^\perp).$$

By using again (1.1) and (2.5) we deduce

$$(\nabla_W h)(J_1\eta, Y) - (\nabla_{J_1\eta} h)(W, Y) = \frac{c}{4}g(J_1W, Y)\eta,$$

and

$$(\nabla_W h)(J_1\eta, Y) - (\nabla_Y h)(W, J_1\eta) = \frac{c}{2}g(J_1W, Y)\eta.$$

Adding the last two relations and taking account of (2.15) we get

$$(\nabla_W h)(J_1\eta, Y) = \frac{c}{4}g(J_1W, Y)\eta,$$

which together with (3.2) implies

$$(3.3) \quad \frac{c}{4}g(J_1W, Y)g(\eta, h(Y, Y)) = 0, \quad \forall Y, W \in \Gamma(D), \eta \in \Gamma(\nu^\perp).$$

Finally, we suppose $c \neq 0$ and $D \neq \{0\}$. Then we take $Y = J_1W$ in (3.3) and by linearity obtain:

$$(3.4) \quad g(h(X, Y), \eta) = 0, \quad \forall X, Y \in \Gamma(D), \eta \in \Gamma(\nu^\perp).$$

As a consequence of (2.6) and (3.4) we deduce $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$, which contradict (2.11). This completes the proof of the theorem.

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