

On the prime power divisors of the iterates of the Euler- φ function

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Abstract. In this paper we prove that for each fixed $k \geq 1$,

$$\Delta(\varphi_k(n)) := \Omega(\varphi_k(n)) - \omega(\varphi_k(n)) = (1 + o(1)) \frac{1}{k} (\log \log x)^k (\log \log \log x)$$

holds for almost all n , and that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \left\{ n \leq x \mid \frac{\Delta(\varphi(n)) - s(x)}{\sqrt{\log \log x (\log \log \log x)}} < z \right\} = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt,$$

where $s(x) = (\log \log x)(\log \log \log x) + c_1 \log \log x + o(\log \log x)$.

1. Introduction

Let $\varphi_k(n) = \varphi(\varphi_{k-1}(n))$ ($\varphi_0(n) = n$, $\varphi_1(n) = \varphi(n)$) be the k -fold iterate of the Euler totient function. Let P be the set of primes, and the letters p, q, π, Q with and without suffixes denote prime numbers. Φ is the Gaussian distribution function. $\omega(n)$ counts the number of distinct prime factors of n , $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. Let $\Delta(n) = \Omega(n) - \omega(n)$. As usual $p^r \parallel n$ means that $p^r \mid n$ but $p^{r+1} \nmid n$.

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For the variable x let $x_1 = \log x$, $x_r = \log x_{r-1}$ ($r = 2, 3, \dots$).
In our recent paper [1] we proved that for each fixed k

$$(1.1) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \# \left\{ n \leq x \mid \frac{\omega(\varphi(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z)$$

$$(1.2) \quad \lim_{\pi(x)} \frac{1}{\pi(x)} \# \left\{ p \leq x \mid \frac{\omega(\varphi(p-1)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z)$$

where $a_k = \frac{1}{(k+1)!}$, $b_k = \frac{1}{k! \sqrt{2k+1}}$.

We are interested in the distribution of $\Delta(\varphi_k(n))$.

Theorem 1. For each fixed $k \geq 1$, for all but $o(x)$ integers $n \leq x$,

$$(1.3) \quad \Delta(\varphi_k(n)) = a_{k-1}(1 + o(1))x_2^k x_4$$

holds.

Remark 1. Theorem 1 implies that (1.1) remains valid if we change ω by Ω . For an arbitrary additive function f and an interval $I \subseteq [1, \infty)$ let

$$f(n \mid I) = \sum_{\substack{q^\beta \parallel n \\ q \in I}} f(q^\beta),$$

and let $f_z(n) = f(n \mid [0, z])$.

We shall prove

Theorem 2. Let

$$s(x) := \sum_{p \leq x} \frac{\Omega_{x_2^2}(p-1)}{p} + \pi(x_2) + \sum_{x_2 \leq q \leq x_2^2} (1 - e^{-x_2/q}) - \omega x_2,$$

where

$$\omega = \int_1^\infty \frac{\exp(-\xi)}{\xi^2} d\xi.$$

Then

$$\lim_{x \rightarrow \infty} \frac{1}{x} \# \left\{ n \leq x \mid \frac{\Delta(\varphi(n)) - s(x)}{\sqrt{x_2 x_4}} < z \right\} = \Phi(z).$$

Remark 2. One can prove that $s(x) = x_2 x_4 + c_1 x_2 + o(x_2)$.

2. Lemmata

Let

$$s(x, D, \ell) := \sum_{\substack{p \leq x \\ p \equiv \ell \pmod{D}}} 1/p.$$

Lemma 1. *Uniformly in $1 \leq D \leq x$ we have*

$$(2.1) \quad s(x, D, 1) \leq \frac{cx_2}{\varphi(D)}.$$

PROOF. From sieve theorems we know that the number of primes $p \equiv 1 \pmod{D}$ in the interval $[aD, 2aD]$ is less than $c_1 \frac{aD}{\log a} \frac{1}{\varphi(D)}$. Applying this for $a = 2, 2^2, 2^3, \dots$ one gets that

$$s(x, D, 1) \leq \sum_{\substack{p < 2D \\ p \equiv 1 \pmod{D}}} \frac{1}{p} + \frac{c_1}{\varphi(D)} \sum_{j=1}^{j_0} \frac{1}{2^j D} \frac{2^j D}{\log 2^j},$$

where j_0 is the largest integer for which $2^{j_0} D \leq x$. The first sum is at most $\frac{1}{D+1}$, the second sum is less than $\frac{c_1}{\varphi(D)} \frac{1}{\log 2} \sum_{j=1}^{j_0} \frac{1}{j} < \frac{c_2}{\varphi(D)} \log j_0 < \frac{c_3}{\varphi(D)} x_2$. Hence (2.1) is immediate. $\square$

Let $U_k(x; D) := \#\{n \leq x, D \mid \varphi_k(n)\}$.

Lemma 2. *For every $k \geq 0, r \geq 0$ there exist numerical values $c(k, r) \leq c(k, r+1)$ for which*

$$U_k(x; D) \leq C(k, \Omega(D)) \frac{xx_2^{k\Omega(D)}}{D}$$

whenever $e^e < x, 1 \leq D \leq x$.

PROOF. The assertion is true for $k = 0$. $C(0, r) = 1$ is a suitable choice. Assume that it is true for $k-1$ instead of k . Let $D = p_1^{a_1} \dots p_r^{a_r}$, and the prime decomposition of $\varphi_{k-1}(n)$ let $\prod \pi^{\delta_\pi}$. Then

$$\varphi_k(n) = \prod \pi^{(\delta_\pi - 1)} \prod (\pi - 1) = E_1 E_2.$$

Let $p_j^{e_j} \parallel E_1, b_j = \min(e_j, a_j), D_1 = \prod_{j=1}^r p_j^{b_j}, D_2 = \prod_{j=1}^r p_j^{a_j - b_j}$.

Assume now that $D \mid \varphi_k(n)$. Then, either $D_2 = 1$, or $D_2 > 1$ and there is a product partition of D_2 as $\xi_1 \dots \xi_s$, $\xi_i > 1$ ($i = 1, \dots, s$) for which $\xi_\ell \mid \pi_\ell - 1$, $\pi_\ell \mid \varphi_{k-1}(n)$ ($\ell = 1, \dots, s$). It is clear that $s \leq \sum_{j=1}^r (a_j - b_j)$. For a fixed choice of $\xi_1, \dots, \xi_s$ let $T = T_s$ run over those numbers $\pi_1, \dots, \pi_s \leq x$ for which $\xi_\ell \mid \pi_\ell - 1$ ($\ell = 1, \dots, s$). Then

$$U_k(x; D) \leq \sum_{D_1 D_2 = D} \sum_s \sum_{\substack{\xi_1, \dots, \xi_s \\ \xi_1, \dots, \xi_s = D_2}} \sum_T U_{k-1}(x, D_1 T).$$

Since $\Omega(D_1 T) = \Omega(D_1) + s \leq \Omega(D_1) + \Omega(D_2) = \Omega(D)$, from our hypothesis we obtain that

$$U_k(x; D) \leq C(k-1, \Omega(D)) x x_2^{(k-1)\Omega(D)} \sum_{D_1 D_2 = D} \sum_s \sum_{\xi_1, \dots, \xi_s} \sum_T \frac{1}{D_1 T}.$$

From Lemma 1 we deduce that the sum $\sum \frac{1}{T}$ is less than $c^s x_2^s / \varphi(\xi_1) \dots \varphi(\xi_s)$. Furthermore $\varphi(\xi_j) \geq \frac{1}{2} \xi_j$. Since the number of solutions of $D = D_1 D_2$ and that of the number of the product partition of D_2 is bounded by a function of $\Omega(D)$, and $s \leq \Omega(D)$, we obtain that the assertion is true for k .

The proof is complete. $\square$

As a consequence we have

Lemma 3. *Let $K_x \rightarrow \infty$ arbitrarily slowly, $\omega_0 = K_x$, $\omega_j = x_2^{2j}$ ($j = 1, 2, \dots$). Let $k \geq 0$ be fixed. Then the number M of that integers $n \leq x$ for which there exists at least one prime $p \geq \omega_k$ such that $p^2 \mid \varphi_k(n)$ is $o(x)$.*

PROOF. It is enough to observe that

$$M \leq \sum_{p \geq \omega_k} U_k(x, p^2), \quad \sum_{p \geq \omega_k} \frac{1}{p^2} \ll \frac{1}{\omega_k x_3}, \quad \square$$

Lemma 4. *For $e^e \leq y \leq x$ we have*

$$\sum_{p \leq x} (\Omega_y(p-1) - \log \log y)^2 \ll \frac{x}{\log x} \log \log y,$$

and

$$\sum_{p \leq x} \frac{1}{p} |\Omega_y(p-1) - \log \log y| \ll x_2 (\log \log y).$$

PROOF. The first, Turán–Kubilius type inequality can be proved by squaring out, applying the Siegel–Walfisz theorem. The second inequality is an easy consequence of the first one. $\square$

3. Proof of Theorem 1

Let the prime decomposition of $\varphi_{k-1}(n)$ be $\prod \pi^{\delta_\pi}$, $y = x_2^{2^k}$. From Lemma 3 we obtain that $\Delta_y(\varphi_k(n)) = \Delta(\varphi_k(n))$ for all but $o(x)$ integers $n \leq x$. Furthermore,

$$\Omega_y(\varphi_k(n)) = \sum_{\substack{\pi \leq y \\ \pi | \varphi_{k-1}(n)}} (\delta_\pi - 1) + \sum_{\pi | \varphi_{k-1}(n)} \Omega_y(\pi - 1).$$

Let $\eta(n)$ denote the second sum. The first sum is $\Delta_y(\varphi_{k-1}(n))$. Thus we have

$$\Delta_y(\varphi_k(n)) = \Delta_y(\varphi_{k-1}(n)) + \eta(n) - \omega_y(\varphi_k(n)).$$

Since

$$\sum_{n \leq x} \omega_y(\varphi_k(n)) = \sum_{p \leq y} U_{k-1}(x; p),$$

from Lemma 2 and from the obvious inequality $U_{k-1}(x; p) \leq x$ we obtain that the right hand side is less than

$$x\pi(x_2^k) + C(k-1, 1)xx_2^k \sum_{x_2^k \leq p \leq y} 1/p \ll xx_2^k.$$

Consequently

$$\frac{1}{x} \#\{n \leq x \mid \omega_y(\varphi_k(n)) \geq x_2^k x_5\} \rightarrow 0 \quad (x \rightarrow \infty).$$

Let $\tilde{\eta}(n) = (\log \log y)\omega(\varphi_{k-1}(n))$. Then

$$|\eta(n) - \tilde{\eta}(n)| \leq \sum_{\pi | \varphi_{k-1}(n)} |\Omega_y(\pi - 1) - \log \log y|,$$

and so by Lemma 2, and Lemma 4

$$\begin{aligned} \sum_{n \leq x} |\eta(n) - \tilde{\eta}(n)| &\leq \sum_{\pi \leq x} |\Omega_y(\pi - 1) - \log \log y| U_{k-1}(x; \pi) \\ &\ll x x_2^{k-1} \sum_{\pi \leq x} |\Omega_y(\pi - 1) - \log \log y| \frac{1}{\pi} \ll x x_2^k (\log \log y)^{1/2}, \end{aligned}$$

Thus

$$\frac{1}{x} \# \left\{ n \leq x \mid |\eta(n) - \tilde{\eta}(n)| > x_2^k x_4^{3/4} \right\} \rightarrow 0 \quad (x \rightarrow \infty).$$

Consequently,

$$\Delta(\varphi_k(n)) = \Delta_y(\varphi_{k-1}(n)) + (\log \log y) \omega(\varphi_{k-1}(n)) + O\left(x_2^k x_4^{3/4}\right)$$

for all but $o(x)$ integers $n \leq x$.

Since $\omega(\varphi_{k-1}(n)) = a_{k-1} x_2^k + O\left(x_2^{k-1/2} x_5\right)$ (see (1.1)) for almost all n , and by induction on k we may assume that $\Delta_y(\varphi_{k-1}(n)) = O\left(x_2^{k-1} x_4\right)$, therefore

$$\Delta(\varphi_k(n)) = a_{k-1} x_2^k x_4 + O\left(x_2^k x_4^{3/4}\right)$$

for almost all n . The proof is complete.

4. Further lemmata

Let $V = \exp(\exp(\sqrt{x_2}))$, $Y = \exp(x_1 e^{-\sqrt{x_2}})$, $u = \frac{x_1}{\log Y}$, $\beta = \log \frac{\log Y}{\log V} = x_2 - 2\sqrt{x_2}$, $J_1 = \left[\frac{x_2}{x_3}, x_2\right]$, $J_2 = [x_2, x_2^2]$, $J = J_1 \cup J_2$, $L = [V, Y]$.

Let

$$\psi_0(n) := \prod_{\substack{p < V \\ p|n}} (p-1), \quad \psi_1(n) := \prod_{\substack{p|n \\ p \in L}} (p-1), \quad \psi_2(n) := \prod_{\substack{p > Y \\ p|n}} (p-1),$$

$\psi(n) := \psi_0(n) \psi_1(n) \psi_2(n)$, $T(n) = \prod_{p|n} p^{\alpha_p(n)-1}$ where $\alpha_p(n)$ is defined as the exponent for which $p^{\alpha_p(n)} \parallel n$.

Let

$$\tau(k) := \prod_{\substack{\pi \in L \\ \pi \equiv 1 \pmod{k}}} (1 - 1/\pi).$$

From the prime number theorem for arithmetical progressions one can get easily that

$$(4.1) \quad \tau(k) = \exp(-\beta/\varphi(k)) \left(1 + O\left(\frac{1}{\varphi(k) \log V}\right) \right),$$

uniformly as $k \in [\sqrt{x_2}, x_2^4]$, say.

Assume from now on that $q, q_1, q_2 \in J$.

Let

$$\varrho(q_1, q_2) := \frac{\tau(q_1)\tau(q_2)}{\tau(q_1 q_2)}.$$

Since $\tau(q_1 q_2) = 1 + O\left(\frac{x_2}{q_1 q_2}\right) = 1 + O\left(\frac{x_3^2}{x_2}\right)$, therefore

$$(4.2) \quad \varrho(q_1, q_2) = \tau(q_1)\tau(q_2) \left(1 + O\left(\frac{x_2}{q_1 q_2}\right) \right).$$

Let

$$(4.3) \quad \mathcal{A}(q) := \sum_{\substack{n \leq x \\ \psi_1(n) \not\equiv 0 \pmod{q}}} 1,$$

$$(4.4) \quad \mathcal{B}(q_1, q_2) := \sum_{\substack{n \leq x \\ \psi_1(n) \equiv 0 \pmod{q_1} \\ \psi_1(n) \not\equiv 0 \pmod{q_2}}} 1,$$

$$(4.5) \quad \mathcal{C}(q_1, q_2) := \sum_{\substack{n \leq x \\ \psi_1(n) \not\equiv 0 \pmod{q_j} \\ j=1,2}} 1.$$

As an immediate consequence of Theorem 2.5 in HALBERSTAM–RICHERT [7], we obtain that

$$(4.6) \quad \mathcal{A}(q) = x\tau(q)(1 + O(e^{-u})),$$

$$(4.7) \quad \mathcal{C}(q_1, q_2) = x\tau(q_1)\tau(q_2) \left(1 + O\left(\frac{x_2}{q_1 q_2}\right) \right).$$

Since $\mathcal{B}(q_1, q_2) + \mathcal{C}(q_1, q_2) = \mathcal{A}(q_2)$, therefore

$$(4.8) \quad \mathcal{B}(q_1, q_2) = x\tau(q_2)(1 - \tau(q_1)) + O\left(\frac{xx_2}{q_1 q_2}\right).$$

Lemma 5. *Let*

$$(4.9) \quad K_1 := \sum_{n \leq x} [\omega(\psi_1(n)|J_2) - A_x]^2,$$

where

$$(4.10) \quad A_x = \sum_{q \in J_2} (1 - \tau(q)).$$

Then

$$(4.11) \quad K_1 \ll xx_2.$$

PROOF. Let $S(x) := \sum_{n \leq x} \omega(\psi_1(n)|J_2)$; $F(x) := \sum_{n \leq x} \omega^2(\psi_1(n)|J_2)$. Then

$$\begin{aligned} S(x) &= \sum_{q \in J_2} (x - \mathcal{A}(q)) + O(x_2^2) = xA_x + O\left(xe^{-u} \sum_{q \in J_2} \tau(q)\right) \\ &= xA_x + O(x), \end{aligned}$$

since $\tau(q) \leq 1$ and $\sum \tau(q) \leq e^u$.

Furthermore, $F(x) = S(x) + \sum_1$, where

$$(4.12) \quad \sum_1 = \sum_{\substack{q_1 \neq q_2 \\ q_1, q_2 \in J_2}} \sum_{\substack{\psi_1(n) \equiv O \pmod{q_j} \\ j=1,2}} 1.$$

The inner sum of the right hand side of (4.12) equals to

$$[x] - \mathcal{B}(q_1, q_2) - \mathcal{B}(q_2, q_1) - \mathcal{C}(q_1, q_2).$$

Thus

$$\begin{aligned} \sum_1 &= [x] \sum_{q_1 \neq q_2} 1 - x \sum_{q \neq q_2} \tau(q_2)(1 - \tau(q_2)) + \tau(q_1)(1 - \tau(q_2)) \\ &\quad - x \sum_{q_1 \neq q_2} \tau(q_1)\tau(q_2) + O(xx_2) = x \sum_{q_1 \neq q_2} (1 - \tau(q_1))(1 - \tau(q_2)) + O(xx_2) \\ &= xA_x^2 + O(xx_2) - x \sum_{q \in J_2} (1 - \tau(q))^2. \end{aligned}$$

Since

$$K_1 = F(x) - 2A_x S(x) + A_x^2[x],$$

we have

$$\begin{aligned} K_1 &= xA_x^2 + O(xx_2) + xA_x + O(x) - 2A_x(xA_x + O(x)) \\ &\quad + O(xA_x) - x \sum_{q \in J_2} (1 - \tau(q))^2 \ll xA_x, \end{aligned}$$

and finally, from

$$A_x \ll \sum \beta/q_1 \ll x_2,$$

we obtain that

$$K_1 \ll xx_2.$$

This completes the proof of Lemma 5.

Let

$$(4.13) \quad \mathcal{B}_x = \sum_{q \in J_1} \tau(q).$$

From (4.1) we obtain that

$$\begin{aligned} \mathcal{B}_x &= \sum_{q \in J_1} e^{-\beta/q-1} \left(1 + O\left(\frac{1}{q(\log V)^c} \right) \right) \\ &= \sum_{q \in J_1} e^{-\beta/q} \left(1 + O\left(\frac{\beta}{q^2} \right) + O\left(\frac{1}{q(\log V)^c} \right) \right). \end{aligned}$$

To estimate the main term

$$T := \sum_{q \in J_1} e^{-\beta/q},$$

we shall use the prime number theorem in the form

$$\Delta(z) := \pi(z) - \ell iz \ll ze^{-c_1 \sqrt{x_3}} \quad \text{for } z \in J_1.$$

Thus

$$T = \int_{J_1} e^{-\beta/\eta} \frac{d\eta}{\log \eta} + \int_{J_1} e^{-\beta/\eta} d\Delta(\eta) = T_1 + T_2.$$

By partial intergation we have

$$T_2 = O\left(x_2 e^{-c_1 \sqrt{x_3}}\right) + \int |\Delta(\eta)| |(e^{\beta/\eta})'| d\eta.$$

The integral on the right hand side is bounded by

$$\ll e^{-c_1 \sqrt{x_3}} \int \eta \left(e^{-\beta/\eta}\right)' d\eta = O\left(x_2 e^{-c_1 \sqrt{x_3}}\right) + e^{-c_1 \sqrt{x_3}} \int_{J_1} e^{-\beta/\eta} d\eta.$$

Since $\frac{1}{\log \eta} = \frac{1}{x_3} + O\left(\frac{x_4}{x_3}\right)$ in $\eta \in J_1$, therefore

$$T_1 = \frac{1}{x_3} \left(1 + O\left(\frac{x_4}{x_3}\right)\right) \int_{J_1} e^{-\beta/\eta} d\eta.$$

Introducing the new variable $\xi = \beta/\eta$ we have

$$(4.14) \quad \int_{J_1} e^{-\beta/\eta} d\eta = \beta \int_{\beta/x_2}^{\beta x_3/x_2} \frac{\exp(-\xi)}{\xi^2} d\xi.$$

Let

$$(4.15) \quad \omega := \int_1^\infty \frac{\exp(-\xi)}{\xi^2} d\xi.$$

The integral on the right hand side of (4.14) equals to

$$\omega + O\left(\frac{1}{\sqrt{x_2}}\right) + O\left(\frac{\exp\left(-\frac{\beta x_3}{x_2}\right)}{x_3^2}\right) = \omega + O\left(\frac{1}{\sqrt{x_2}}\right),$$

consequently

$$T_1 = \beta \omega + O(\sqrt{x_2}),$$

and

$$T_2 = O\left(x_2 e^{-x_1 \sqrt{x_3}}\right).$$

One can prove similarly that

$$\sum_{q \in J_1} e^{-\beta/q} \left(O\left(\frac{\beta}{q^2}\right) + O\left(\frac{1}{q(\log V)^c}\right) \right) = O(1).$$

Consequently

$$(4.16) \quad \mathcal{B}_x = \omega\beta + O(\sqrt{x_2}) = \omega x_2 + O(\sqrt{x_2}).$$

Let $h(n)$ be the number of those primes q in J_1 for which $q \nmid \psi_1(n)$.

Lemma 6. *Let*

$$(4.17) \quad K_2 := \sum_{n \leq x} (h(n) - \mathcal{B}_x)^2,$$

Then

$$(4.18) \quad K_2 = O(xx_2).$$

PROOF. From (4.6), (4.7) we infer that

$$(4.19) \quad U(x) := \sum_{n \leq x} h(n) = \sum_{q \in J_1} \mathcal{A}(q) = x(1 + O(e^{-u}))\mathcal{B}_x.$$

$$(4.20) \quad V(x) := \sum_{n \leq x} h^2(n) = U(x) + \sum_2,$$

where

$$\begin{aligned} \sum_2 &= \sum_{n \leq x} h(n)(h(n) - 1) = \sum_{q_1, q_2 \in J_1, q_1 \neq q_2} \mathcal{C}(q_1, q_2) \\ (4.21) \quad &= x(1 + O(e^{-u})) \sum_{q_1 \neq q_2, q_1, q_2 \in J_1} \varrho(q_1, q_2) \\ &= x(1 + O(e^{-u})) \sum_{q_1, q_2 \in J_1, q_1 \neq q_2} \tau(q_1)\tau(q_2) \left(1 + O\left(\frac{\beta}{q_1 q_2}\right)\right). \end{aligned}$$

Consequently

$$(4.22) \quad \sum_2 = x \left(\mathcal{B}_x^2 - \sum_{q \in J_1} \tau^2(q) + O(e^{-u} \mathcal{B}_x^2) + O\left(x_2 \left(\sum_{q \in J_1} \frac{\tau(q)}{q} \right)^2 \right) \right).$$

Since

$$K_2 = U(x) + \sum_2 - 2\mathcal{B}_x U(x) + [x]\mathcal{B}_x^2,$$

therefore, by (4.19), (4.20), (4.22) we get that

$$K_2 = O(x\mathcal{B}_x) + O(e^{-u}x\mathcal{B}_x^2) + O\left(x\left(\sum_{q \in J_1} \tau^2(q) + x_2\left(\sum_{q \in J_1} \frac{\tau(q)}{q}\right)^2\right)\right).$$

From (4.16), $\mathcal{B}_x \ll x_2$, furthermore

$$\sum_{q \in J_1} \tau^2(q) \ll \frac{x_2}{x_3}, \quad \sum_{q \in J_1} \frac{\tau(q)}{q} \ll \sum_{q \in J_1} \frac{1}{q} = o(1),$$

thus $K_2 \ll xx_2$.

The proof of the lemma is finished. $\square$

Let H be the number of primes in J_1 . Then, by Lemma 6

$$\sum_{n \leq x} (\omega(\psi_1(n)|J_1) - (H - \mathcal{B}_x))^2 \ll xx_2,$$

consequently

$$(4.23) \quad \frac{1}{x} \#\left\{n \leq x, |\omega(\psi_1(n)|J_1) - (H - \mathcal{B}_x)| > x_5\sqrt{x_2}\right\} \ll \frac{1}{x_5^2}.$$

Similarly, from Lemma 5,

$$(4.24) \quad \frac{1}{x} \#\left\{n \leq x, |\omega(\psi_1(n)|J_2) - A_x| > x_5\sqrt{x_2}\right\} \ll \frac{1}{x_5}.$$

5. The distribution of $\Omega_{x_2^2}(\psi(n))$

Let

$$(5.1) \quad A(x) = \sum_{p \leq x} \frac{\Omega_{x_2^2}(p-1)}{p}, \quad \mathcal{B}^2(x) = \sum_{p \leq x} \frac{\Omega_{x_2^2}^2(p-1)}{p}.$$

We observe that

$$f(n) := \Omega_{x_2^2}(\psi(n)) = \sum_{p|n} \Omega_{x_2^2}(p-1) = \sum_{p|n} f(p)$$

is a strongly additive function depending on the parameter x . To prove that

$$(5.2) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \# \left\{ n \leq x, \Omega_{x_2^2}(\psi(n)) - A(x) \leq z\mathcal{B}(x) \right\} = \Phi(z),$$

we can apply Theorem 12.15 in ELLIOTT [8], and the Berry–Esseen theorem (see Lemma 1.48 in [8]). The conditions of these theorems are satisfied. Indeed, by using the prime number theorem for arithmetical progressions and Lemma 1, we can deduce easily that

$$(5.3) \quad A(x) = x_2 x_4 + O(x_2), \quad \mathcal{B}^2(x) = (1 + o(1))x_2 x_4^2,$$

and that

$$(5.4) \quad \sum_{p \leq x} \frac{\Omega_{x_2^2}^3(p-1)}{p} \ll x_2 x_4^3.$$

(5.3), (5.4) imply that

$$\sum_{\substack{x^\epsilon < p \leq x \\ |f(p)| > \lambda \mathcal{B}(x)}} \frac{1}{p} \rightarrow 0$$

for arbitrary positive constants ϵ and λ , thus the condition of Theorem 12.15 holds. Since the left hand side of (5.4) is $o(\mathcal{B}^3(x))$, therefore the remainder term in the Berry–Esseen theorem is $o(1)$, thus (5.2) holds.

6. Proof of Theorem 2

It is clear that for $\ell = 1, 2$:

$$(6.1) \quad \begin{aligned} \omega(\psi_1(n)|J_\ell) &\leq \omega(\psi(n)|J_\ell) \\ &\leq \omega(\psi_1(n)|J_\ell) + \omega(\psi_0(n)|J_\ell) + \omega(T(n)|J_\ell). \end{aligned}$$

Let

$$\sum_0 = \sum_{n \leq x} \omega(\psi_0(n)|J), \quad \sum_1 = \sum_{n \leq x} \omega(T(n)|J), \quad \sum_2 = \sum_{n \leq x} \omega(\psi_2(n)|J).$$

First we estimate $\sum_1$. We have

$$\sum_1 \ll \sum_{q \in J} \frac{x}{q^2} \ll x.$$

Similarly,

$$\begin{aligned} \sum_2 &\ll \sum_{q \in J} \sum_{\substack{n \leq x \\ \psi_2(n) \equiv O \pmod{q}}} 1 \ll \sum_{q \in J} \sum_{\substack{p \equiv 1 \pmod{q} \\ Y < p \leq x}} x/p \\ &\ll x \log \frac{\log x}{\log Y} \sum_{q \in J} 1/q \ll x \sqrt{x_2}. \end{aligned}$$

Finally

$$\begin{aligned} \sum_0 &\ll \sum_{q \in J} \sum_{\substack{n \leq x \\ \psi_0(n) \equiv O \pmod{q}}} 1 \ll x \sum_{q \in J} \sum_{\substack{q|p-1 \\ p \leq V}} 1/p \\ &\ll x \sqrt{x_2} \sum_{q \in J} \frac{1}{q} = O(x \sqrt{x_2}). \end{aligned}$$

We deduced that

$$\sum_0 + \sum_1 + \sum_2 = O(x \sqrt{x_2}).$$

Consequently

$$(6.2) \quad \frac{1}{x} \# \left\{ n \leq x \mid |\omega(\psi(n)|J) - \omega(\psi_1(n)|J)| > x_5 \sqrt{x_2} \right\} \rightarrow 0.$$

We can observe that

$$\frac{1}{x} \# \left\{ n \leq x \mid |\Delta(\varphi(n)) - \Delta(\psi(n))| \geq K_x \right\} \rightarrow 0,$$

if K_x is an arbitrary function which tends to infinity. Let $K_x = x_5$. Therefore it is enough to prove the theorem for $\psi(n)$ instead of $\varphi(n)$. From Lemma 3 we obtain that the asymptotic density of the integers $n \leq x$ for which $p^2 \mid \psi(n)$ for some $p > x_2^2$ is zero. Thus $\Delta(\psi(n)) = \Delta_{x_2^2}(\psi(n))$ for all but $o(x)$ integers up to x . By sieve theorems one can deduce that the

number of the integers $n \leq x$ for which there is a prime $q \leq x_2/x_3$ such that $q \nmid \psi(n)$ is less than

$$cx \sum_{q < x_2/x_3} \prod_{\substack{\pi \equiv 1 \pmod{q} \\ \pi < x}} \left(1 - \frac{1}{\pi}\right) \leq cx \sum_{q < x_2/x_3} \exp\left(-\frac{x_2}{q-1}\right) = o(x).$$

Thus, removing no more than $o(x)$ integers $n \leq x$, for the others

$$(6.3) \quad \omega_{x_2/x_3}(\psi(n)) = \pi(x_2/x_3).$$

Hence, and from (5.2) we obtain that

$$\omega_{x_2^2}(\psi(n)) = \pi(x_2/x_3) + \omega(\psi_1(n)|J) + O(x_5\sqrt{x_2})$$

for all but $o(x_2)$ integers $n \leq x$. By using (4.23) and (4.24) we get that

$$(6.4) \quad \omega_{x_2^2}(\psi(n)) = \pi(x_2) + A_x - \mathcal{B}_x + O(x_5\sqrt{x_2})$$

for all but $o(x)$ integers $n \leq x$.

By (5.2) we have that

$$\frac{1}{x} \# \left\{ n \leq x \mid \frac{\Delta(\varphi(n)) - A(x) - \pi(x_2) - A_x + \mathcal{B}_x}{\sqrt{x_2 x_4}} < z \right\} \rightarrow \Phi(z) \quad \text{as } x \rightarrow \infty.$$

From (4.1), (4.10) we can deduce easily that

$$A_x = \sum_{x_2 \leq q \leq x_2^2} \left(1 - \exp\left(-\frac{x_2}{q}\right)\right) + O(\sqrt{x_2}).$$

Then, by (4.16) our theorem follows.

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