

Connections and regularity on the cotangent bundle

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Abstract. Considering some diffeomorphisms between the tangent and the cotangent bundles of a smooth manifold M obtained from regular geometric objects on TM and T^*M we establish some relations between mechanical structures on TM and T^*M . We construct a mechanical structure on T^*M and we also study the condition under which a regular vector field defining a mechanical structure on T^*M is a semi-Hamilton or Hamilton vector field.

Introduction

There are some essential differences between the differential geometries of the tangent and cotangent bundles of a differentiable manifold M , although the bundles are isomorphic, as vector bundles.

The tangent bundle has a naturally defined integrable almost tangent structure J and one can define the notion of a semispray (second order differential equation vector field) on it. From a semispray one can derive easily a (nonlinear) connection on the tangent bundle. The cotangent bundle T^*M has a naturally defined symplectic structure ω but one cannot define naturally something similar to the notion of almost tangent structure or semispray.

In [1] it was introduced the notion of regularity and that of adapted almost tangent structure on the cotangent bundle T^*M , used for the study of some geometric properties of the tangent and cotangent bundles, considering some diffeomorphism between them. A regular vector field K on

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T^*M defines an integrable adapted almost tangent structure and a connection Γ_K on T^*M . On the other hand, this regular vector field defines naturally an M -bundle diffeomorphism Φ from T^*M to TM which transfers some results from the differential geometry of the cotangent bundle to that of the tangent bundle, and conversely. Using a regular vector field (which always exists) one defines the notion of mechanical structure on T^*M , which generalizes the notion of Hamiltonian system on T^*M and is correlated with the notion of mechanical system on TM . A regular vector field K on T^*M defining a mechanical structure is used to define another two symmetric connections Γ'_K and Γ''_K on T^*M . In this paper we study the conditions for the existence of regular vector fields K' and K'' on T^*M , with the same associated M -vector field (see [11]) like K , such that the associated connections to K' and K'' , $\Gamma_{K'}$ and $\Gamma_{K''}$ are just Γ'_K and Γ''_K .

In [2], based on a similar idea, considering a regular 1-form on TM we have constructed a semispray which defines a connection on TM . The above regular 1-form defines naturally an M -bundle diffeomorphism Ψ from TM to T^*M , of the Legendre transformation type (see [4], [6]), which transfers some results from the differential geometry of the tangent bundle to that of the cotangent bundle, and conversely.

In [3] we established some relations between regular 1-forms and mechanical systems on TM . Considering the diffeomorphism Ψ from TM to T^*M determined by a regular 1-form defining a mechanical systems on TM , we construct in this paper a mechanical structure on T^*M , and also, we study the conditions under which the regular vector field defining a mechanical structure on T^*M is a semi-Hamilton or Hamilton vector field.

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1. Preliminaries

We begin by describing the geometric structures of the tangent and cotangent bundles TM and T^*M of a smooth n -dimensional manifold M . First, we shall present, in short, some result included in [1], [2] and [3]. Denote by $\tau : TM \longrightarrow M$, $\pi : T^*M \longrightarrow M$ the natural projections and let (U, x^i) , $i = 1, \dots, n$; be a local chart on M . Then the local chart $(\tau^{-1}(U); x^i, y^i)$ is induced on TM where $x^i = x^i \circ \tau$ (by abuse of notation) and y^i are the vector space coordinates with respect to the natural local

frame $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ in TM defined by (U, x^i) . Similarly the local chart $(\pi^{-1}(U); q^i, p_i)$ is induced on T^*M where $q^i = x^i \circ \pi$ and p_i are the vector space coordinates with respect to the natural local frame $(dx^1, \dots, dx^n)$ in T^*M defined by (U, x^i) .

Due to the special form of the change rules of the induced local charts on TM (on T^*M), it is possible to define the notion of an M -tensor field on these bundles (see [11]).

Definition. An M -tensor field of type (k, l) on TM (on T^*M) is defined by sets of local coordinate components $T_{j_1 \dots j_l}^{i_1 \dots i_k}$, assigned to every induced local chart on TM (on T^*M) such that the change rule, when a change of induced local charts on TM (on T^*M) is performed, is the same with the change rule of the local coordinate components of a tensor field of type (k, l) on M .

Remark that any M -tensor field on TM may be thought of as an ordinary tensor field T with the expression

$$T = T_{j_1 \dots j_l}^{i_1 \dots i_k} \frac{\partial}{\partial y^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}.$$

However, there are many other possibilities to interpret an M -tensor field as an ordinary tensor field on TM . Remark also that any ordinary tensor field on the base manifold may be thought of as an M -tensor field on TM , having the same type and with the components in the induced local chart on TM , equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold. The M -tensor field associated to a covariant tensor field on the base manifold M may be thought of as the pull back of this tensor field by the smooth submersion $\tau : TM \longrightarrow M$.

Consider the natural integrable almost tangent structure on TM , defined by the tensor field J of type $(1, 1)$ on TM having the local coordinate expression

$$J = \frac{\partial}{\partial y^i} \otimes dx^i.$$

In the case of the cotangent bundle the notion of *adapted almost tangent structure* is defined ([1]) by a tensor field J on T^*M of type $(1, 1)$ such that

$\text{Ker } J = \text{Im } J = VT^*M$, where $VT^*M = \text{Ker } \pi_*$ is the integrable vertical distribution. The local coordinate expression of J must be

$$(1) \quad J = g_{ij} \frac{\partial}{\partial p_i} \otimes dq^j,$$

where the components $g_{ij}(q, p)$ define a nondegenerate M -tensor field of type $(0, 2)$ on T^*M . The integrability condition for the adapted almost tangent structure consists in the existence of a regular M -vector field $\xi^i(q, p)$ (an M -tensor field of type $(1, 0)$) on T^*M such that $g^{ij} = \frac{\partial \xi^j}{\partial p_i}$, where $g^{ij}(q, p)$ are the entries of the inverse of the matrix $(g_{ij}(q, p))$, i.e. $g_{ij}g^{ik} = g_{ji}g^{ki} = \delta_j^k$, and the matrix $(g_{ij}(q, p))$ is nondegenerate.

The components $\xi^i(q, p)$ of an M -vector field on T^*M may be thought of as the last n components of an M -bundle (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$. So, the local coordinate expression of Φ is

$$(2) \quad x^i = q^i; \quad y^i = \xi^i(q, p).$$

The inverse Φ^{-1} of Φ has the local coordinate expression

$$(3) \quad q^i = x^i; \quad p_i = \zeta_i(x, y).$$

The components $\zeta_i(x, y)$ define an $M - 1$ -form on TM .

Let K be a vector field on T^*M , with the local coordinate expression

$$(4) \quad K(q, p) = \xi^i(q, p) \frac{\partial}{\partial q^i} + \chi_i(q, p) \frac{\partial}{\partial p_i}.$$

*The components ξ^i define an M -vector field on T^*M , called the associated M -vector field of K .*

The vector field K is called regular if its associated M -vector field is regular, i.e. the matrix $(\frac{\partial \xi^j}{\partial p_i})$ is nondegenerate.

From a regular vector field K (from its associated M -vector field) one can derive an integrable adapted almost tangent structure J_ξ , with the local coordinate expression (1), and a nonlinear connection Γ_K , (thought of as an almost product structure on T^*M such that the eigendistribution corresponding to the eigenvalue -1 is the vertical distribution) given by

$$(5) \quad \Gamma_K = -\mathcal{L}_K J_\xi$$

Consider the canonical 1-form (the Liouville form) $\theta = p_i dq^i$ on T^*M and let $\omega = d\theta$ be the canonical symplectic structure on T^*M .

Recall that a vector field $H = \xi^i \frac{\partial}{\partial q^i} + \chi_i \frac{\partial}{\partial p_i}$ on T^*M is called a Hamilton vector field if it is regular and $\mathcal{L}_H \omega = 0$. This condition is expressed locally by the relations

$$(6) \quad (i) \frac{\partial \xi^i}{\partial p_j} = \frac{\partial \xi^j}{\partial p_i}; \quad (ii) \frac{\partial \chi_i}{\partial p_j} = -\frac{\partial \xi^j}{\partial q^i}; \quad (iii) \frac{\partial \chi_i}{\partial q^j} = \frac{\partial \chi_j}{\partial q^i}.$$

Definition. A vector field K on T^*M defines a mechanical structure on T^*M if it is regular and the matrix $(g^{ij}) = (\frac{\partial \xi^j}{\partial p_i})$ defined by its associated M -tensor field is symmetric (see [1], [4]).

In this case only the condition (6) (i) is fulfilled. There exists a smooth real function $h : T^*M \rightarrow \mathbb{R}$ such that $\xi^i = \frac{\partial h}{\partial p_i}$ but, generally, $\chi_i \neq -\frac{\partial h}{\partial q^i}$. Consider the form $\eta = i_K \omega - (-dh)$. So $i_K \omega = -dh + \eta$. Therefore $\chi_i = -\frac{\partial h}{\partial q^i} + \eta_i$.

Then $\eta = \eta_i dq^i$ is a *semibasic 1-form* on T^*M . ($\eta(V) = 0$ whenever V is a vertical vector field on T^*M .)

Definition. The vector field K on T^*M is a semi-Hamilton vector field if it is regular and $\mathcal{L}_K \omega$ is a semibasic 2-form on T^*M , i.e. $i_V(\mathcal{L}_K \omega) = 0$, whenever V is a vertical vector field on T^*M .

This condition may be expressed locally by (6) (i) and (6) (ii). In the case of a vector field K defining a mechanical structure on T^*M one defines in [1] another two connections on T^*M . First, one defines the symmetric tensor field g of type $(0, 2)$ on T^*M , by

$$g(Z_1, Z_2) = -\omega(Z_1, J_\xi Z_2); \quad Z_1, Z_2 \in \chi(T^*M).$$

Its local coordinate expression is $g = g_{ij} dq^i \otimes dq^j$. Then the nonlinear connection Γ'_K on T^*M is defined by the condition

$$\omega(Z_1, \Gamma'_K Z_2) = (\mathcal{L}_K g)(Z_1, Z_2); \quad Z_1, Z_2 \in \chi(TM).$$

Another connection Γ''_K on T^*M defined by a vector field K defining a mechanical structure is obtained in a similar way from the tensor field $\tilde{g} = g_{ij} \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial p_j}$. Both connections Γ'_K and Γ''_K are symmetric.

Recall that a *semispray* (a second order differential vector field) on the tangent bundle TM (see [4], [6], [7], [10]) is a vector field S on TM

such that $JS = C$, where C is the Liouville vector field on TM (the local coordinate expression is $C = y^i \frac{\partial}{\partial y^i}$).

The local coordinate expression of a semispray is $S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x, y) \frac{\partial}{\partial y^i}$. There is a connection Γ (thought of as an almost product structure) on TM (see [4], [6], [7], [10]) derived from the semispray S by

$$(7) \quad \Gamma = -\mathcal{L}_S J.$$

The connection Γ is called *the canonical connection associated to the semispray S* .

In [1] it is proved the following result: *Let be K a regular vector field on T^*M and assume that its associated M -vector field defines a global M -bundle diffeomorphism $\Phi : T^*M \longrightarrow TM$. Then the vector field Φ_*K is a semispray on TM whose derived connection Γ is just the image by Φ of the connection Γ_K defined by (5).*

In [2] it is considered a regular 1-form on TM from which one derives, by using the integrable almost tangent structure and the Liouville vector field on TM , a semispray on TM . This semispray has its associated connection. Then, the above regular 1-form defines naturally an M -bundle (local) diffeomorphism Ψ from TM to T^*M by which we can establish some correspondences between the differential geometry of these bundles.

Let α be a 1-form on TM with local coordinate expression

$$(8) \quad \alpha = \alpha_j(x, y) dy^j + \beta_j(x, y) dx^j.$$

The components α_j define an $M - 1$ -form on TM , called *the associated $M - 1$ -form of α* . The form α is called *regular* if the matrix with the entries $(g_{ij} = \frac{\partial \alpha_i}{\partial y^j})$ is nondegenerate. Assuming that the 1-form α is regular, we consider the 1-form θ on TM defined by

$$(9) \quad \theta = \alpha \circ J = i_J \alpha,$$

where i_J is the vertical derivation on TM . The 1-form θ corresponds to the Cartan 1-form defined by a regular Lagrangian on TM (see [7]) and has the local coordinate expression

$$\theta = \alpha_i(x, y) dx^i.$$

The 2-form $d\theta = di_J\alpha$ is a symplectic form. Consider the 1-form β on TM defined by

$$(10) \quad \beta = di_C\alpha - \alpha,$$

and define the vector field S on TM by

$$(11) \quad i_S d\theta = -\beta.$$

We find that S is a semispray on TM . Its associated connection is given by (7). Just like in the case of a regular M -vector field, the components $\alpha_i(x, y)$ of α may be thought as the last n components of an M -bundle map $\Psi : TM \longrightarrow T^*M$. This is the (local) diffeomorphism mentioned above. The local coordinate expression of Ψ is

$$(12) \quad q^i = x^i; \quad p_i = \alpha_i(x, y),$$

where $(\pi^{-1}(U); q^i, p_i)$ is the induced local chart on T^*M . The inverse Ψ^{-1} of Ψ has the local coordinate expression

$$(13) \quad x^i = q^i; \quad y^i = u^i(q, p).$$

The central result of [2] is the following:

*If α is a regular 1-form on TM , there is a semispray S on TM derived from it by (9), (10), (11). Assuming that its associated $M-1$ -form defines a global M -bundle diffeomorphism $\Psi : TM \longrightarrow T^*M$, the vector field Ψ_*S is a regular vector field on T^*M whose derived connection Γ_K is the image by Ψ of the connection Γ associated to the semispray S .*

2. Regular 1-forms and mechanical structures on TM and T^*M

In [3], some relations between regular 1-forms and regular mechanical systems on TM were studied. In this section, considering the diffeomorphism $\Psi : TM \longrightarrow T^*M$ defined by a regular 1-form α , defining a mechanical system on TM , we shall construct a mechanical structure on T^*M . Then, we shall study the condition under which the regular vector field defining a mechanical structure on T^*M is a semi-Hamilton vector field.

Recall that a *mechanical system on TM* (see [4], [9]) is a triple $m = (M, F, \rho)$, where $F : TM \rightarrow \mathbb{R}$ is a smooth real valued function and ρ is a semibasic 1-form on TM (i.e. $\rho(V) = 0$, for any vertical vector field V on TM).

The *vertical differentiation* d_J on TM (see [4], [7]) is a derivation of $\Lambda(M)$ of degree one, defined by

$$d_J = [i_J, d] = i_J d - di_J.$$

Consider the form $\omega_F = dd_J F$. If ω_F is a symplectic form (an equivalent condition is for the Hessian $(\frac{\partial^2 F}{\partial y^i \partial y^j})$ of F to be nondegenerate) the mechanical system m is said to be *regular*. In this case, the vector field S defined by

$$(14) \quad i_S \omega_F = -dE_F + \rho,$$

where $E_F = CF - F$ is the *energy function associated to F* , is a semispray (see [4], [7], [8], [9]).

m is said to be *conservative* if ρ is a closed form. In this case the mechanical system is Lagrangian. In [3] it is considered a regular 1-form $\alpha = \alpha_i(x, y)dy^i + \beta_i(x, y)dx^i$ such that its associated $M - 1$ -form defined by α_i satisfies the conditions

$$(15) \quad \frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}; \quad i, j = 1, \dots, n. \text{ (i.e. } g_{ij} = g_{ji}).$$

Then (15) may be expressed by

$$\alpha_i = \frac{\partial F}{\partial y^i},$$

where F is a smooth real valued function on TM .

Considering $\rho_i = \beta_i - \frac{\partial F}{\partial x^i}$, the form $\rho = \rho_i dx^i$ is a semibasic 1-form and $\alpha = dF + \rho$. The triple $m = (M, F, \rho)$ is proved to be a regular mechanical system and conversely, if $m = (M, F, \rho)$ is a regular mechanical system, then $\alpha = dF + \rho$ is a regular 1-form and its associated $M - 1$ -form satisfies (15).

Theorem 1. *If the differential form α on TM is regular and $\frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}$; $i = 1, \dots, n$, then the (local) diffeomorphism $\Psi : TM \longrightarrow T^*M$ given by (12) defines a mechanical structure $K = \Psi_*S$ on T^*M .*

PROOF. By a straightforward computation we get

$$K = \Psi_*S = u^i(q, p) \frac{\partial}{\partial q^i} + \chi_i(q, p) \frac{\partial}{\partial p_i};$$

From the condition for Ψ^{-1} to be the inverse of Ψ , we get $\frac{\partial u^j}{\partial p_i} \circ \Psi = g^{ij}$. It follows

$$\frac{\partial u^j}{\partial p_i} \circ \Psi = g^{ij} = g^{ji} = \frac{\partial u^i}{\partial p_j} \circ \Psi;$$

Therefore K defines a mechanical structure on T^*M . $\square$

Proposition 2. *If the differential form α on TM is regular and $\frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}$, $i, j = 1, \dots, n$, then Ψ_*S is a semi-Hamilton vector field on T^*M if and only if the form ρ is a basic 1-form on TM .*

*In other words, η is a basic 1-form on T^*M if and only if ρ is a basic 1-form on TM .*

PROOF. From the condition for Ψ^{-1} to be the inverse of Ψ , we also get

$$\frac{\partial u^j}{\partial q^k} \circ \Psi = -g^{ij} \frac{\partial \alpha_j}{\partial x^k}.$$

Ψ_*S is a semi-Hamiltonian vector field if and only if the condition (6) (ii) is fulfilled. In this case $\xi^i = u^i$ and $\chi_i = \beta_i \circ \Psi^{-1}$. Then the condition (6) (ii) becomes

$$\begin{aligned} \frac{\partial u^j}{\partial q^i} + \frac{\partial \beta_j}{\partial y^k} \frac{\partial u^k}{\partial p_j} &= 0 \quad \Leftrightarrow \quad -g^{kj} \frac{\partial \alpha_k}{\partial x^i} + \frac{\partial \beta_i}{\partial y^k} g^{jk} = 0 \quad \Leftrightarrow \\ g^{jk} \left(\frac{\partial \alpha_k}{\partial x^i} - \frac{\partial \beta_i}{\partial y^k} \right) &= 0 \quad \Leftrightarrow \quad \frac{\partial \alpha_k}{\partial x^i} = \frac{\partial \beta_i}{\partial y^k} \quad \Leftrightarrow \quad \frac{\partial}{\partial y^k} \left(\frac{\partial F}{\partial x^i} - \beta_i \right) = 0 \\ &\Leftrightarrow \quad \frac{\partial \rho_i}{\partial y^k} = 0. \end{aligned}$$

The last conditions means that the form ρ is a basic 1-form on TM .

Proposition 3. *Under the hypothese of Proposition 1, the mechanical system defined by α on TM is conservative (Lagrangian) if and only if the mechanical system defined by $K = \Psi_*S$ is conservative (Hamiltonian).*

*In other words, ρ is a closed form on TM if and only if η is a closed form on T^*M .*

PROOF. The mechanical system defined by $K = \Psi_*S$ is conservative if and only if $d\eta = 0$, so $\mathcal{L}_K\omega = 0$. Taking into account the previous result, we have that (6) (iii) is equivalent with $\frac{\partial\beta_i}{\partial x^j} = \frac{\partial\beta_j}{\partial x^i}$. But $\beta_i = \frac{\partial F}{\partial x^i} + \rho_i$. Therefore $\frac{\partial\beta_i}{\partial x^j} = \frac{\partial\beta_j}{\partial x^i}$ if and only if $\frac{\partial\rho_i}{\partial x^j} = \frac{\partial\rho_j}{\partial x^i}$ and, since ρ is already a basic form this means that ρ is a closed form on TM . $\square$

3. Conditions for existence of regular geometric objects on T^*M

Consider now another problem related to the above correspondences between the tangent and the cotangent bundles. Assume that K is a regular vector field on T^*M and let $\beta_i(x, y)$; $i = 1, \dots, n$; be n -smooth real valued function on the domain $\tau^{-1}(U)$ of an induced local chart on TM . We ask for the local existence of a regular 1-form $\alpha = \alpha_j dy^j + \beta_j dx^j$ (of a regular $M - 1$ -form α_i) such that Φ_*K is just the semispray derived from α .

Theorem 4. *Consider a regular vector field K on T^*M with the local coordinate expression (4) and let $\beta_i(x, y)$; $i = 1, \dots, n$ be n -smooth real valued functions on the domain $\tau^{-1}(U)$ of an induced local chart on TM . If the ordinary differential system on $\tau^{-1}(U)$*

$$\begin{aligned}
 (16) \quad & \frac{dx^1}{y^1} = \dots = \frac{dx^n}{y^n} \\
 & = \frac{dy^1}{\frac{\partial \xi^1}{\partial q^k}(x, \zeta(x, y))y^k + \frac{\partial \xi^1}{\partial p_k}(x, \zeta(x, y))\chi_k(x, \zeta(x, y))} = \dots \\
 & \dots = \frac{dy^n}{\frac{\partial \xi^n}{\partial q^k}(x, \zeta(x, y))y^k + \frac{\partial \xi^n}{\partial p_k}(x, \zeta(x, y))\chi_k(x, \zeta(x, y))} = \dots \\
 & \dots = \frac{d\alpha_1}{\beta_1} = \dots = \frac{d\alpha_n}{\beta_n}
 \end{aligned}$$

has n first integrals $C^1, \dots, C^n(x, y, \alpha)$ such that

$$(17) \quad \frac{D(C_1, \dots, C_n)}{D(y^1, \dots, y^n)} \neq 0; \quad \frac{D(C_1, \dots, C_n)}{D(\alpha_1, \dots, \alpha_n)} \neq 0;$$

then there exists a regular 1-form $\alpha = \alpha_j dy^j + \beta_j dx^j$ (so, with the last n components β_i) such that the semispray $\Phi_* K$ derives from α .

PROOF. Consider the (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$ and its inverse $\Phi^{-1} : TM \longrightarrow T^*M$ having the local coordinate expression (2) and (3), respectively. The semispray S derived from α has the local coordinate expression

$$S = y^i \frac{\partial}{\partial x^i} + g^{ki} (\beta_k - \frac{\partial \alpha_k}{\partial x^l} y^l) \frac{\partial}{\partial y^i}.$$

The relations $\Phi_* K = S$ are equivalent with

$$g^{ji} (\beta_j - \frac{\partial \alpha_j}{\partial x^k} y^k) = y^j (\frac{\partial \xi^i}{\partial q^j} \circ \Phi^{-1}) + (\chi_j g^{ji}) \circ \Phi^{-1}.$$

From these relations transvected by $g_{ki} = \frac{\partial \alpha_k}{\partial y^i}$ we get the quasilinear partial differential system

$$(18) \quad \frac{\partial \alpha_i}{\partial x^j} y^j + \frac{\partial \alpha_i}{\partial y^j} \left[y^k \frac{\partial \xi^j}{\partial q^k} (x, \zeta(x, y)) + \frac{\partial \xi^j}{\partial p_k} (x, \zeta(x, y)) \chi_k(x, \zeta(x, y)) \right] = \beta_i$$

whose characteristic system is just (16). The first integrals $C_1, \dots, C_n$ assure, using the implicit function theorem, the obtaining of the components $\alpha_i(x, y)$. $\square$

Remark. Under the hypothesis of the existence of this regular $M - 1$ -form α_i (of a regular form α) let us consider the (local) diffeomorphism $\Psi : TM \longrightarrow T^*M$, defined by the $M - 1$ -form α_i . Then $\Psi = \Phi^{-1}$ if and only if

$$\alpha_i(x, y) = \zeta_i(x, y), \quad i = 1, \dots, n.$$

Using the relations (18) we obtain the following result:

Proposition 5. Let $K(q, p) = \xi^i(q, p) \frac{\partial}{\partial q^i} + \chi_i(q, p) \frac{\partial}{\partial p_i}$ be a regular vector field on T^*M and let $\beta_i(x, y); i = 1, \dots, n;$ be n smooth real valued functions on the domain $\tau^{-1}(U)$ of an induced local chart on TM . Consider the (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$ defined by K . There exists a regular 1-form $\alpha = \alpha_j dy^j + \beta_j dx^j$ (so, with the last n components β_i) such that the semispray Φ_*K is derived from α and Ψ is just Φ^{-1} (in this case $\Psi_*S = K$) if and only if the components $\zeta_i(x, y)$ satisfy the conditions:

$$(19) \quad \frac{\partial \zeta_i}{\partial x^j} y^j + \frac{\partial \zeta_i}{\partial y^j} \left[y^k \frac{\partial \xi^j}{\partial q^k}(x, \zeta(x, y)) + \frac{\partial \xi^j}{\partial p_k}(x, \zeta(x, y)) \chi_k(x, \zeta(x, y)) \right] = \beta_i.$$

Let us consider a regular vector field $K(q, p) = \xi^i(q, p) \frac{\partial}{\partial q^i} + \chi_i(q, p) \frac{\partial}{\partial p_i}$ defining a mechanical structure on T^*M . From K one can derive the connection Γ_K (its associated connection) and, also, the connections Γ'_K and Γ''_K , as above. We ask for the existence of a regular vector field K' on T^*M , with the same associated M -vector field (ξ^i) such that $\Gamma'_K = \Gamma_{K'}$, that is Γ'_K is just the connection associated to K' . Then, we shall consider the same problem for Γ''_K . For the study of these problems we shall use the following result of [1]:

If the components $\xi^i(q, p); i = 1, \dots, n$ define a regular vector field on T^*M and Γ^* is a connection on T^*M then there exists a regular vector field K on T^*M with its associated M -vector field ξ^i and such that $\Gamma^* = \Gamma_K$ if and only if the following integrability conditions are fulfilled

$$(20) \quad \frac{\delta g_{kj}}{\delta q^i} - \frac{\delta g_{ki}}{\delta q^j} = g_{hi} \frac{\partial \Gamma_{kj}}{\partial p_h} - g_{hj} \frac{\partial \Gamma_{ki}}{\partial p_h},$$

where $\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - \Gamma_{ji} \frac{\partial}{\partial p_j}$, $i = 1, \dots, n$ are local vector field defining a local frame in the horizontal distribution HT^*M , defined by Γ^* .

Theorem 6. Let $K(q, p) = \xi^i(q, p) \frac{\partial}{\partial q^i} + \chi_i(q, p) \frac{\partial}{\partial p_i}$ be a regular vector field defining a mechanical structure on T^*M . There exists a regular vector field K' with the same associated M -vector field (ξ^i) such that $\Gamma'_K = \Gamma_{K'}$ if and only if the following integrability condition are fulfilled:

$$(21) \quad \left(\frac{\partial g_{ki}}{\partial p_h} g_{lj} - \frac{\partial g_{kj}}{\partial p_h} g_{li} \right) \left(\frac{\partial \xi^l}{\partial q^h} + \frac{\partial \chi_h}{\partial p_l} \right) = 0.$$

PROOF. The coefficients Γ'_{ij} of the connection Γ'_K are given by

$$\Gamma'_{ij} = \frac{1}{2} \left\{ Kg_{ij} + g_{ik} \frac{\partial \xi^k}{\partial q^j} + g_{kj} \frac{\partial \xi^k}{\partial q^i} \right\}.$$

The result is then proved by a straightforward computation by using the relations (20). $\square$

Remark. If K is a semi-Hamilton vector field, the expression in the second parenthesis vanishes. Therefore, the relations (21) are fulfilled in this case.

Theorem 7. Let $K(q, p) = \xi^i(q, p) \frac{\partial}{\partial q^i} + \chi_i(q, p) \frac{\partial}{\partial p_i}$ be a regular vector field defining a mechanical structure on T^*M . There exists a regular vector field K'' with the same associated M -vector field (ξ^i) , such that $\Gamma''_K = \Gamma_{K''}$ if and only if the following integrability condition are fulfilled:

$$(22) \quad \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ki}}{\partial q^j} = g_{kl} \left(g_{hj} \frac{\partial^2 \chi_i}{\partial p_h \partial p_l} - g_{hi} \frac{\partial^2 \chi_j}{\partial p_h \partial p_l} \right).$$

PROOF. One uses the relations (20) where the coefficients Γ''_{ij} of the connection Γ''_K are given by

$$\Gamma''_{ij} = \frac{1}{2} \left\{ Kg_{ij} - g_{ik} \frac{\partial \chi_j}{\partial p_k} - g_{kj} \frac{\partial \chi_i}{\partial p_k} \right\}.$$

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