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On a class of means of two variables

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To Professor Béla Gyires on his 90th birthday

Abstract. Let $CM(I)$ denote the class of all continuous and strictly monotonic real functions defined on the interval I. Let $L : I^2 \to I$ be a fixed mean on I. A mean $M : I^2 \to I$ is called an L-conjugate mean on I if there exists $\varphi \in CM(I)$ for which $M(x,y) = L^*_\varphi(x,y) := \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(L(x,y)))$ holds for all $x, y \in I$. We solve the following problems for L-conjugate means: equality, comparison, determining homogeneous and translative means and inequalities involving them. Furthermore, we examine when such a mean is quasi-arithmetic, if $L = A$, where A is the arithmetic mean.

1. Introduction

Let $I \subset \mathbb{R}$ be an open interval. A function $M : I^2 \to I$ is called a *mean* in I if it satisfies the following properties:

$$
(1.1)\qquad
\begin{aligned}
&\text{(i)} && \text{If } x, y \in I \text{ and } x \neq y \text{ then} \\
& && \min\{x,y\} < M(x,y) < \max\{x,y\}; \\
&\text{(ii)} && M(x,y) = M(y,x) \text{ for all } x, y \in I; \\
&\text{(iii)} && M \text{ is continuous on } I^2.
\end{aligned}
$$

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Let $CM(I)$ denote the class of all *continuous* and *strictly monotonic* real functions defined on the interval I. We remind of the following

Definition 1. A mean $M : I^2 \to I$ is called *quasi-arithmetic* in I if there exists $\varphi \in CM(I)$ such that

$$M(x,y) = A_\varphi(x,y) := \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) \tag{1.2}$$

for all $x, y \in I$. Then the function φ is called the *generating function* of the quasi-arithmetic mean A_φ ([3], [4], [6], [15]).

In the theory of quasi-arithmetic means answers are known for the following problems: (p1) problem of equality ([6], [9], [12]); (p2) problem of comparison ([6], [8], [10], [14]); (p3) determining homogenous quasi-arithmetic means and inequalities involving them ([6], [16], [5], [10]); (p4) determining translative quasi-arithmetic means and inequalities involving them; (p5) characterization problem ([1], [2], [3], [4], [5], [7], [11], [13], [16], [19], [20]).

Inspired by the paper [15], we define a new class of means of two variables, and answer problems (p1), (p2), (p3), and (p4) for this class. Furthermore, we examine when such a mean is quasi-arithmetic. This new class of means is defined the following way:

Definition 2. Let $L : I^2 \to I$ be a fixed mean on I. A mean $M : I^2 \to I$ is called L-conjugate mean in I if there exists $\varphi \in CM(I)$ for which

$$M(x,y) = L^*_\varphi(x,y) := \varphi^{-1}(\varphi(x)+\varphi(y)-\varphi(L(x,y))) \tag{1.3}$$

for all $x, y \in I$. Then the function φ is called the generating function of the L-conjugate mean L^*_φ.

It can easily be seen that for any $\varphi \in CM(I)$ $M = L^*_\varphi : I^2 \to I$ is a mean in I, that is, the properties (1.1) are fulfilled [15].

2. Equality and comparison

Let $L : I^2 \to I$ be a fixed mean in I. The *problem of equality* (type p1) for L-conjugate means is the following: What conditions are necessary and sufficient for a pair of functions $\varphi, \psi \in CM(I)$ in order that

$$L^*_\varphi(x,y) = L^*_\psi(x,y) \tag{2.1}$$

should hold for all $x, y \in I$? This problem will also be solved if we examine the apparently more difficult problem of comparison (type p2): What conditions are necessary and sufficient for a pair of functions $\varphi, \psi \in CM(I)$ in order that

$$L^*_\varphi(x,y) \le L^*_\psi(x,y) \tag{2.2}$$

should hold for all $x, y \in I$? The latter question is answered by

Theorem 1. *Let $\varphi, \psi \in CM(I)$. Then the inequality (2.2) holds for all $x, y \in I$ if and only if*

(i) $\psi \circ \varphi^{-1}$ *is convex on the interval $\varphi(I) =: J$ for increasing ψ, or*

(ii) $\psi \circ \varphi^{-1}$ *is concave on the interval $\varphi(I) =: J$ for decreasing ψ.*

Proof. We prove (i), the proof of (ii) is similar. So let $\psi \in CM(I)$ be *increasing*. Then (2.2) implies

$$\psi \circ \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(L(x,y))) \le \psi(x) + \psi(y) - \psi(L(x,y))$$

for all $x, y \in I$. From this, with notations $\varphi(x) =: u$, $\varphi(y) =: v$ ($u, v \in \varphi(I) = J$) and $f := \psi \circ \varphi^{-1}$ ($f \in CM(J)$), we have

$$f(u + v - M(u,v)) + f(M(u,v)) \le f(u) + f(v), \tag{2.3}$$

where

$$M(u,v) := \varphi(L(\varphi^{-1}(u), \varphi^{-1}(v))) \qquad (u, v \in J) \tag{2.4}$$

is a mean in J. We need the following lemma (see also [15], [17]).

Lemma 1. *Let $M : J^2 \to J$ be a mean on the open interval $J \subset \mathbb{R}$. Then the sequence defined by equations $M_1(u,v) := M(u,v)$ and $M_{n+1}(u,v) := M(M_n(u,v), u + v - M_n(u,v))$ ($n \in \mathbb{N}$; $u, v \in J$) is convergent and*

$$\lim_{n\to\infty} M_n(u,v) = \frac{u+v}{2}. \tag{2.5}$$

Proof. If $M : J^2 \to J$ is a mean in J, then the function $J^2 \ni (u,v) \mapsto u+v-M(u,v)$ is also a mean, thus the sequence $M_n(u,v)$ ($n \in \mathbb{N}$)

is well-defined. If $u = v$, the assertion clearly holds, since $M(u,u) = u$ $(u \in J)$.

Let $u < v$ $(u, v \in J)$ be fixed. It can easily be seen that for the closed intervals

$$I_k := [\alpha_k(u,v), \omega_k(u,v)],$$

with the notations

$$\alpha_k(u,v) := \min\{M_k(u,v),\ u + v - M_k(u,v)\},$$
$$\omega_k(u,v) := \max\{M_k(u,v),\ u + v - M_k(u,v)\}$$

we have $I_{k+1} \subset I_k$ $(k \in \mathbb{N})$, and, moreover, the symmetry of M implies

$$M_{k+1}(u,v) = M(\alpha_k(u,v), \omega_k(u,v)), \tag{2.6}$$

since $\frac{1}{2}(\alpha_k(u,v), \omega_k(u,v)) = \frac{u+v}{2}$, $\frac{u+v}{2} \in \bigcap_{k=1}^{\infty} I_k$ holds.

Let $\sup_{k\in\mathbb{N}} \alpha_k(u,v) = \lim_{k\to\infty} \alpha_k(u,v) = \alpha(u,v)$ and $\inf_{k\in\mathbb{N}} \omega_k(u,v) = \lim_{k\to\infty} \omega_k(u,v) = \omega(u,v)$, then

$$\alpha_l(u,v) \le \alpha(u,v) \le \omega(u,v) \le \omega_s(u,v) \tag{2.7}$$

for all $l, s \in \mathbb{N}$. We show that $\alpha(u,v) = \omega(u,v) = \frac{u+v}{2}$. If there existed $u < v$ such that $\alpha(u,v) < \omega(u,v)$ then by the property of means

$$\alpha(u,v) < M(\alpha(u,v), \omega(u,v)) < \omega(u,v)$$

would hold. On the other hand, the continuity of M and $(\alpha_k(u,v), \omega_k(u,v)) \to (\alpha(u,v), \omega(u,v))$ $(k \to \infty)$ imply the existence of $N \in \mathbb{N}$ for which

$$\alpha(u,v) < M(\alpha_N(u,v), \omega_N(u,v)) < \omega(u,v),$$

that is, by (2.6),

$$M_{N+1}(u,v) \in \left]\alpha(u,v), \omega(u,v)\right[.$$

Now $M_{n+1}(u,v)$ equals either $\alpha_{N+1}(u,v)$ or $\omega_{N+1}(u,v)$, which contradicts (2.7). Thus $\alpha(u,v) = \omega(u,v)$ is the only number that belongs to $\bigcap_{k=1}^{\infty} I_k$, that is, $\alpha(u,v) = \omega(u,v) = \frac{u+v}{2}$. So from (2.6) we have (2.5). □

Now we continue the proof of Theorem 1.

From inequality (2.3), using the notations of Lemma 1,

$$f(u+v-M_n(u,v))+f(M_n(u,v))\le f(u)+f(v) \tag{2.8}$$

follows for all $n\in\mathbb{N}$ and $u,v\in J$, which can be proved by induction. For $n=1$, (2.8) holds by (2.3). If (2.8) holds for n, then putting $M_n(u,v)$ for u and $u+v-M_n(u,v)$ for v in (2.8), we have by the assumption

$$\begin{aligned}&f(u+v-M_{n+1}(u,v))+f(M_{n+1}(u,v))\\&\qquad\le f(u+v-M_n(u,v))+f(M_n(u,v))\le f(u)+f(v).\end{aligned}$$

Using the assertion of the lemma, since $f\in CM(J)$, with $n\to\infty$, from (2.8) we obtain

$$2f\left(\frac{u+v}{2}\right)\le f(u)+f(v) \tag{2.9}$$

for all $u,v\in J$, that is, f is Jensen-convex in J. Since f is continuous, f is convex in J [14]. This proves the necessity of the condition.

Now suppose that $\psi\in CM(I)$ is increasing and $f:=\psi\circ\varphi^{-1}$ is convex on the interval $\varphi(I)=J$. Let $u,v\in J$ be arbitrary. Then there exists $0<\lambda<1$ such that for the mean $M:J^2\to J$ defined in (2.4)

$$M(u,v)=\lambda u+(1-\lambda)v$$

holds. Thus, by the convexity of f

$$\begin{aligned}&f(u+v-M(u,v))+f(M(u,v))\\&\quad=f(u+v-\lambda u-(1-\lambda)v)+f(\lambda u+(1-\lambda)v)\\&\quad=f((1-\lambda)u+\lambda v)+f(\lambda u+(1-\lambda)v)\\&\quad\le(1-\lambda)f(u)+\lambda f(v)+\lambda f(u)+(1-\lambda)f(v)\\&\quad=f(u)+f(v).\end{aligned}$$

From this inequality, with the substitution $\varphi(x)=u$, $\psi(y)=v$ ($x,y\in I$ are arbitrary) and from (2.4), by $M(u,v)=\varphi(L(x,y))$ we have

$$\begin{aligned}&\psi\circ\varphi^{-1}(\varphi(x)+\varphi(y)-\varphi(L(x,y)))+\psi\circ\varphi^{-1}(\varphi(L(x,y)))\\&\qquad\le\psi(x)+\psi(y),\end{aligned}$$

which implies, since ψ^{-1} is increasing, $L^*_\varphi(x,y)\le L^*_\psi(x,y)$ for all $x,y\in I$. $\square$

Theorem 2. *Let $\varphi, \psi \in CM(I)$. The equality (2.1) holds for all $x, y \in I$ if and only if there exist real constants $\alpha \neq 0$ and β such that*

$$\psi(x) = \alpha\varphi(x) + \beta \tag{2.10}$$

for all $x \in I$.

Proof. It can easily be seen that $L^*_{\psi}(x,y) = L^*_{-\psi}(x,y)$ for all $x, y \in I$, thus, by Theorem 1, both $\psi \circ \varphi^{-1} =: f$ and $-\psi \circ \varphi^{-1} = -f$ are convex in $\varphi(I) =: J$, that is, for all values of $u, v \in J$ and $0 < \lambda < 1$

$$f(\lambda u + (1-\lambda)v) = \lambda f(u) + (1-\lambda)f(v).$$

This implies $f(u) = \alpha u + \beta$ $(u \in J)$, where $\alpha \neq 0$ and β are constants. With the notation $u = \varphi(x)$ $(x \in I)$ we obtain (2.10). Conversely, if ψ is of the form (2.10) one can easily check equality (2.1). □

Definition 3. Let $\varphi, \psi \in CM(I)$. ψ and φ are called *equivalent* if there exist real numbers $\alpha \neq 0$ and β for which (2.10) holds for all $x \in I$. Notation: $\psi \sim \varphi$ or $\psi(x) \sim \varphi(x)$ $(x \in I)$.

Theorem 3. *If $\varphi, \psi \in CM(I)$ and $\psi \sim \varphi$ then $A_\varphi = A_\psi$ and $L^*_\varphi = L^*_\psi$, that is, equivalent generating functions define the same quasi-arithmetic or L-conjugate mean.*

Proof. It is known for quasi-arithmetic means [6]. For L-conjugate means, it follows from Theorem 2. □

3. Homogeneous L-conjugate means in the case of homogeneous L

Definition 4. If $\mathbb{R}_+$ denotes the set of positive real numbers and $M : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a mean on $\mathbb{R}_+$, then this mean is called *homogeneous* if

$$M(tx, ty) = tM(x,y) \tag{3.1}$$

holds for all $x, y, t \in \mathbb{R}_+$.

Theorem 4. *Let $L : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a fixed homogeneous mean on $\mathbb{R}_+$ and let $\varphi \in CM(\mathbb{R}_+)$. Then the L-conjugate mean $L_\varphi^* : \mathbb{R}_+^2 \to \mathbb{R}_+$ on $\mathbb{R}_+$ is homogeneous if and only if*

$$\varphi(x) = \frac{x^a - 1}{a} \qquad (a \in \mathbb{R},\ a \neq 0) \tag{3.2}$$

or

$$\varphi(x) = \log x \tag{3.3}$$

for all $x \in \mathbb{R}_+$, up to equivalence for the generating functions. According to this, the L-conjugate homogeneous means for homogeneous L are the family of means of one parameter ($a \in \mathbb{R}$)

$$L_a^*(x,y) := \begin{cases} (x^a + y^a - L(x,y)^a)^{\frac{1}{a}} & \text{if } a \neq 0 \\ \dfrac{xy}{L(x,y)} & \text{if } a = 0. \end{cases} \tag{3.4}$$

Notice that

$$\lim_{a \to 0} L_a^*(x,y) = L_0^*(x,y) \tag{3.5}$$

for all $x, y \in \mathbb{R}_+$.

Proof. Let $L : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a homogeneous mean and $\varphi \in CM(\mathbb{R}_+)$ for which

$$L_\varphi^*(tx, ty) = t L_\varphi^*(x,y) \tag{3.6}$$

holds for all $t, x, y \in \mathbb{R}_+$. For a fixed $t \in \mathbb{R}_+$, let

$$\psi_t(x) := \varphi(tx) \qquad (x \in \mathbb{R}_+). \tag{3.7}$$

Clearly, $\psi_t \in CM(\mathbb{R}_+)$ and

$$\begin{aligned} L_{\psi_t}^*(x,y) &= \psi^{-1}(\psi_t(x) + \psi_t(y) - \psi_t(L(x,y))) \\ &= \frac{1}{t}\varphi^{-1}(\varphi(tx) + \varphi(ty) - \varphi(tL(x,y))) \\ &= \frac{1}{t}\varphi^{-1}(\varphi(tx) + \varphi(ty) - \varphi(L(tx,ty))) \\ &= \frac{1}{t}L_\varphi^*(tx,ty) = L_\varphi^*(x,y) \end{aligned}$$

for all $x, y \in \mathbb{R}_+$. Thus by Theorem 2 there exist real numbers $\alpha(t) \neq 0$ and $\beta(t)$ such that

$$\psi_t(x) = \alpha(t)\varphi(x) + \beta(t)$$

for all $x \in \mathbb{R}_+$, which implies

$$\varphi(tx) = \alpha(t)\varphi(x) + \beta(t) \tag{3.8}$$

for all elements $x \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ and $\alpha(t) \neq 0$. Since $\varphi \in CM(\mathbb{R})$, by a well-known theorem of Lebesgue [18] there exists $x_0 \in \mathbb{R}_+$ at which φ is differentiable. Then the left hand side of (3.8) is differentiable at the point $x = x_0$, i.e., $\varphi'(tx_0)$ exists. Since $tx_0 = s$ runs through all the elements of $\mathbb{R}_+$, $\varphi'(s)$ exists for all $s \in \mathbb{R}_+$. We show that $\varphi'(s) \neq 0$ for all $s \in \mathbb{R}_+$. If there existed $s_0 \in \mathbb{R}_+$ for which $\varphi'(s_0) = 0$, then differentiating (3.8) with respect to x and putting $x = s_0$ we would obtain $\varphi'(ts_0) = 0$ for all $t \in \mathbb{R}_+$, i.e., φ would be constant, which contradicts $\varphi \in CM(\mathbb{R}_+)$.

Thus $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is differentiable and $\varphi'(x) \neq 0$ for all $x \in \mathbb{R}_+$. We look for solutions φ for which $\varphi(1) = 0$ and $\varphi'(1) = 1$. This causes no loss of generality, as for all φ the generating function

$$\varphi^*(t) := \frac{\varphi(t)}{\varphi'(1)} - \frac{\varphi(1)}{\varphi'(1)}$$

satisfies the required property and $\varphi \sim \varphi^*$.

Putting $x = 1$ in equation (3.8), we obtain $\beta(t) = \varphi(t)$. Differentiating equation (3.8) with respect to x we have

$$\varphi'(tx)t = \alpha(t)\varphi'(x),$$

which implies, with the substitution $x = 1$, $\alpha(t) = t\varphi'(t)$. Now putting the results we got back into equation (3.8) and interchanging the variables t and x we get

$$\varphi(tx) = t\varphi'(t)\varphi(x) + \varphi(t) = x\varphi'(x)\varphi(t) + \varphi(x),$$

from which, with the substitution $t = 2$, since $\varphi(2) \neq 0$,

$$x\varphi'(x) - 1 = \frac{2\varphi'(2) - 1}{\varphi(2)}\varphi(x) = a\varphi(x) \tag{3.9}$$

follows, where $a \in \mathbb{R}$ is a constant value. If $a \neq 0$, then the only solution of the differential equation (3.9) is (3.2). If $a = 0$ we obtain solution (3.3). The remaining statement of the theorem is obvious. □

Remark. The functional equation (3.8) and its solutions are known (see [6], p. 69). Here we gave a different argument by using the monotonicity of φ and reducing (3.8) directly to the differential equation (3.9).

The comparison theorem implies

Theorem 5. *If $L : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a homogeneous mean on $\mathbb{R}_+$ and $a, b \in \mathbb{R}_+$ then*

$$L_a^*(x,y) \le L_b^*(x,y) \tag{3.10}$$

holds for all $x, y \in \mathbb{R}_+$ if and only if

$$a \le b. \tag{3.11}$$

Proof. With the notation

$$\chi_a(u) := \begin{cases} \dfrac{u^a - 1}{a} & \text{if } a \neq 0 \\ \log u & \text{if } a = 0 \end{cases} \qquad (u \in \mathbb{R}_+), \tag{3.12}$$

(3.10) holds if and only if

$$L_{\chi_a}^*(x,y) \le L_{\chi_b}^*(x,y) \tag{3.13}$$

holds for all $x, y \in \mathbb{R}_+$, where $\chi_a, \chi_b \in CM(\mathbb{R}_+)$. One can easily check that $\chi_a : \mathbb{R}_+ \to \mathbb{R}$ is differentiable and $\chi_a'(u) > 0$ for all $u \in \mathbb{R}_+$. Therefore, by the comparison theorem, (3.13) holds if and only if $f := \chi_b \circ \chi_a^{-1}$ is *convex* on the interval $J := \chi_a(\mathbb{R}_+)$. Since the function f is differentiable, this holds if and only if

$$f(x) - f(y) \ge (x - y) f'(y)$$

for all $x, y \in J$, that is,

$$\chi_b \circ \chi_a^{-1}(x) - \chi_b \circ \chi_a^{-1}(y) \ge (x - y)\chi_b' \circ \chi_a^{-1}(y) \frac{1}{\chi_a' \circ \chi_a^{-1}(y)},$$

from which, with the notations $u := \chi_a^{-1}(x),\ v := \chi_a^{-1}(y)$ $(u, v \in \mathbb{R}_+)$

$$\frac{\chi_b(u) - \chi_b(v)}{\chi_b'(v)} \ge \frac{\chi_a(u) - \chi_a(v)}{\chi_a'(v)} \tag{3.14}$$

follows for all $u, v \in \mathbb{R}_+$. Putting the function (3.12) into the inequality (3.14) we obtain

$$v\chi_a\left(\frac{u}{v}\right) \leq v\chi_b\left(\frac{u}{v}\right),$$

which implies, with the notation $\dfrac{u}{v} =: s \in \mathbb{R}_+$,

$$\chi_a(s) \leq \chi_b(s) \tag{3.15}$$

for all $s \in \mathbb{R}_+$. Therefore it is enough to show that (3.15) holds for all $s \in \mathbb{R}_+$ if and only if $a \leq b$. This means that $\mathbb{R} \ni a \mapsto \chi_a(s)$ is increasing for any fixed $s \in \mathbb{R}_+$ and there exists $s_0 \in \mathbb{R}_+$ for which $\mathbb{R} \ni a \mapsto \chi_a(s_0)$ is strictly increasing. This follows from

$$\frac{\partial}{\partial a}\chi_a(s) = \begin{cases} \dfrac{s^a \log s^a - s^a + 1}{a^2} & \text{if } a \neq 0 \\[2ex] \dfrac{(\log s)^2}{2} & \text{if } a = 0 \end{cases} \qquad (s \in \mathbb{R}_+)$$

and since $z \log z - z + 1 \geq 0$ $(z \in \mathbb{R}_+)$,

$$\frac{\partial}{\partial a}\chi_a(s) \geq 0 \qquad (s \in \mathbb{R}_+),$$

that is, $a \mapsto \chi_a(s)$ is increasing in a with s fixed. If $s \neq 0$ then $a \mapsto \chi_a(s)$ is strictly increasing. This completes the proof of the theorem. □

As a special case of the theorem we obtain the inequality $L^*_{-1} \leq L^*_0 \leq L^*_1$: If $L : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a homogeneous mean then

$$\left(\frac{1}{x} + \frac{1}{y} - \frac{1}{L(x,y)}\right)^{-1} \leq \frac{xy}{L(x,y)} \leq x + y - L(x,y)$$

for all values $x, y \in \mathbb{R}_+$. Of course, this inequality can be proved in an elementary way.

4. Translative L-conjugate means in the case of translative L

Definition 5. A mean $M : \mathbb{R}^2 \to \mathbb{R}$ on $\mathbb{R}$ is called *translative* if

$$M(t + x, t + y) = t + M(x, y) \tag{4.1}$$

holds for all $t, x, y \in \mathbb{R}$.

Theorem 6. *Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a translative mean in $\mathbb{R}$ and $\varphi \in CM(\mathbb{R})$. Then the L-conjugate mean $L^*_\varphi : \mathbb{R}^2 \to \mathbb{R}$ on $\mathbb{R}$ is translative if and only if*

$$\varphi(x) = \frac{e^{ax}-1}{a} \qquad (a \in \mathbb{R},\ a \neq 0) \tag{4.2}$$

or

$$\varphi(x) = x \tag{4.3}$$

for all $x \in \mathbb{R}$ up to equivalence for the generating functions.

According to this, the L-conjugate translative means for translative L are the family of means of one parameter ($a \in \mathbb{R}$)

$$L^*_{[a]}(x,y) := \begin{cases} \dfrac{1}{a}\log\left(e^{ax}+e^{ay}-e^{aL(x,y)}\right) & \text{if } a \neq 0 \\ x+y-L(x,y) & \text{if } a = 0 \end{cases} \tag{4.4}$$

for which

$$\lim_{a \to 0} L^*_{[a]}(x,y) = L^*_{[0]}(x,y) \tag{4.5}$$

for all $x, y \in \mathbb{R}$.

Proof. Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a translative mean on $\mathbb{R}$ and $\varphi \in CM(\mathbb{R})$. Then

$$H(u,v) := \exp L(\log u, \log v) \qquad (u, v \in \mathbb{R}_+) \tag{4.6}$$

is a *homogeneous* mean on $\mathbb{R}_+$ and with the notation $\psi(u) := \varphi(\log u)$ $(u \in \mathbb{R}_+)$ $\psi \in CM(\mathbb{R}_+)$, furthermore, for all $s, u, v \in \mathbb{R}_+$

$$\begin{aligned} H^*_\psi(su,sv) &= \psi^{-1}(\psi(su)+\psi(sv)-\psi(H(su,sv))) \\ &= \exp\varphi^{-1}(\varphi(\log su)+\varphi(\log sv)-\varphi(L(\log su, \log sv))) \\ &= \exp L^*_\varphi(\log s+\log u, \log s+\log v))) \\ &= \exp(\log s)\exp L^*_\varphi(\log u, \log v) \\ &= sH^*_\psi(u,v), \end{aligned} \tag{4.7}$$

that is, $H^*_\psi : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a homogeneous mean. Thus from Theorem 4 either $\psi(u) = \dfrac{u^a-1}{a}$ $(a \neq 0)$ or $\psi(u) = \log u$ $(u \in \mathbb{R}_+)$ follows, which implies (4.2) and (4.3) for φ. The remaining statement of the theorem is obvious. □

Theorem 7. *If $L : \mathbb{R}^2 \to \mathbb{R}$ is a translative mean on $\mathbb{R}$ and $a, b \in \mathbb{R}$, then*

$$L^*_{[a]}(x, y) \le L^*_{[b]}(x, y) \tag{4.8}$$

holds for all $x, y \in \mathbb{R}$ if and only if

$$a \le b. \tag{4.9}$$

Proof. The statement easily follows from the relation between homogeneous and translative means and Theorem 5. □

5. A-conjugate means which are quasi-arithmetic means

The best-known mean is the arithmetic mean $A : I^2 \to I$ defined by

$$A(x, y) := \frac{x + y}{2} \qquad (x, y \in I) \tag{5.1}$$

and can be defined on any open interval $I \subset \mathbb{R}$. The following problem seems to be natural: For which $\varphi \in CM(I)$ will the A-conjugate mean $A^*_\varphi : I^2 \to I$ be also quasi-arithmetic on the interval I? This means that if φ is the required generating function, then there exists $\psi \in CM(I)$ such that

$$A^*_\varphi(x, y) = A_\psi(x, y) \tag{5.2}$$

holds for all $x, y \in I$. In more detail, for the unknown functions $\varphi, \psi \in CM(I)$ the functional equation

$$\varphi^{-1}\left(\varphi(x) + \varphi(y) - \varphi\left(\frac{x + y}{2}\right)\right) = \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) \tag{5.3}$$

holds for all $x, y \in I$.

The problem has not been solved yet in its most general form. If we require that the generating function φ in (5.2) (or (5.3)) satisfy further conditions then the generating functions φ and ψ can be determined, and therefore we can derive means from them that are A-conjugate and quasi-arithmetic at the same time. Our aim, which will be realized in Section 6, is to prove the following:

Theorem 8. *Suppose that $\varphi \in CM(I)$ is twice differentiable in I. Then the A-conjugate mean $A^*_\varphi : I^2 \to I$ is quasi-arithmetic in I if and only if φ is one of the functions below (disregarding the equivalence of generating functions):*

(i) $\varphi(x) = x$ *if* $x \in I$,

(ii) *There exists* $\lambda \in P_+ := \{\lambda \in \mathbb{R} \mid I + \lambda \subset \mathbb{R}_+\}$ *such that* $\varphi(x) = \log(x + \lambda)$ *if* $x \in I$,

(iii) *There exists* $\mu \in P_- := \{\mu \in \mathbb{R} \mid -I + \mu \subset \mathbb{R}_+\}$ *such that* $\varphi(x) = \log(-x + \mu)$ *if* $x \in I$.

Remarks.

(1) If $I =]a, b[$ $(a, b \in \mathbb{R},\ a < b)$ is bounded then

$$P_+ = \{\lambda \in \mathbb{R} \mid \lambda > -a\} \quad \text{and} \quad P_- = \{\mu \in \mathbb{R} \mid \mu > b\}.$$

(2) If I is not bounded then $I = \mathbb{R}$ and cases (ii) and (iii) do not occur; or $I =]-\infty, b[$ $(b \in \mathbb{R})$ and case (ii) does not occur; or $I =]a, \infty[$ $(a \in \mathbb{R})$ and case (iii) does not occur.

6. The proof of Theorem 8

To make the proof of Theorem 8 easier to read we first prove the following two lemmas:

Lemma 2. *Let $\varphi \in CM(I)$ be (once) continuously differentiable in I and $\varphi'(x) \neq 0$ if $x \in I$. If there exists $\psi \in CM(I)$ for which*

$$A^*_\varphi(x, y) = A_\psi(x, y) \tag{6.1}$$

holds for all $x, y \in I$ then ψ is (once) differentiable in I and $\psi'(x) \neq 0$ if $x \in I$.

Proof. By (6.1), for all $x, y \in I$

$$\psi(x) = 2\psi(A^*_\varphi(x, y)) - \psi(y). \tag{6.2}$$

Now let $x_0 \in I$ be arbitrarily fixed. Then from (6.2)

$$\psi(x_0) = 2\psi(A^*_\varphi(x_0, y)) - \psi(y)$$

follows for all $y \in I$. Since the function $y \mapsto A_\varphi^*(x_0, y)$ $(y \in I)$ is continuous, $A_\varphi^*(x_0, I)$ is a nonvoid open interval. Then by the monotonity of ψ and Lebesgue's theorem ψ is almost everywhere differentiable, thus there exists $y_0 \in I$ such that ψ is differentiable at $A_\varphi^*(x_0, y_0)$. Since φ is differentiable and $\varphi'(x) \neq 0$ $(x \in I)$, its inverse is also differentiable, therefore the function $x \mapsto A_\varphi^*(x, y_0)$ $(x \in I)$ is differentiable at x_0, and by

$$\psi(x) = 2\psi(A_\varphi^*(x, y_0)) - \psi(y_0) \qquad (x \in I)$$

ψ is differentiable at x_0 by the differentiation rule of composite functions. Thus differentiating (6.2) with respect to x we have

$$\psi'(x) = 2\psi'(A_\varphi^*(x, y)) \frac{\varphi'(x) - \varphi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\varphi'(A_\varphi^*(x, y))} \tag{6.3}$$

for all $x, y \in I$.

Now let $x_0 \in I$ be arbitrarily fixed. Then the function $y \mapsto \varphi'(x_0) - \varphi'\left(\frac{x+y}{2}\right)\frac{1}{2}$ $(y \in I)$ takes the value $\varphi'(x_0)\frac{1}{2} \neq 0$ at $y = x_0$, thus, by the continuity of φ, there exists $\delta > 0$ such that for any $y \in]x_0 - \delta, x_0 + \delta[\subset I$, $\varphi'(x_0) - \varphi'\left(\frac{x+y}{2}\right)\frac{1}{2} \neq 0$. On the other hand, as ψ is strictly monotonic, there exists $y_0 \in]x_0 - \delta, x_0 + \delta[$ for which $\psi'(A_\varphi^*(x_0, y_0)) \neq 0$. This implies, by (6.3),

$$\psi'(x_0) \neq 0,$$

which completes the proof of the Lemma. □

Lemma 3. *Let $J \subset \mathbb{R}$ be an open interval and $F : J \to \mathbb{R}$ such that*

$$(F(x) - F(y))\left(F\left(\frac{x+y}{2}\right) - \frac{F(x) + F(y)}{2}\right) = 0 \tag{6.4}$$

holds for all $x, y \in J$. If F is continuous on J then there exist constants $\alpha, \beta \in \mathbb{R}$ for which

$$F(x) = \alpha x + \beta \quad \text{if } x \in I. \tag{6.5}$$

Proof. If F is constant in J then (6.4) holds and $\alpha = 0$ in (6.4). If F is not constant in J then there exist $a, b \in J$ with $a < b$ such that

$F(a) \neq F(b)$. Then (6.4) implies

$$F\left(\frac{a+b}{2}\right) = \frac{F(a)+F(b)}{2}.$$

It can be seen by induction that

$$F\left(\frac{ka+lb}{2^n}\right) = \frac{kF(a)+lF(b)}{2^n},$$

where $k \geq 0$, $l \geq 0$ are integers and $k+l = 2^n$ $(n = 0, 1, 2, \dots)$. From this, by the continuity of F and the density of $\dfrac{ka+lb}{2^n}$ in $[a, b]$, we obtain

$$F(x) = \alpha x + \beta \quad \text{if } x \in [a, b], \tag{6.6}$$

where

$$\alpha = \frac{F(b)-F(a)}{b-a} \neq 0 \quad \text{and} \quad \beta = \frac{bF(a)-aF(b)}{b-a}.$$

If $t \in J$ and $t \notin [a, b]$ and $t \in]a-\delta, a[$ or $t \in]b, b+\delta[$ (where $0 < \delta < \frac{b-a}{2}$) then there exists $x \in]a, b[$ such that $F(x) \neq F(t)$ and $\frac{x+t}{2} \in]a, b[$, from which, by (6.4)

$$\alpha\frac{x+t}{2} + \beta = F\left(\frac{x+t}{2}\right) = \frac{F(x)+F(t)}{2} = \frac{\alpha x + \beta + F(t)}{2}$$

follows, that is, $F(t) = \alpha t + \beta$. Thus the solution (6.6) can be extended to J, with this the proof of the Lemma is complete. □

Proof of Theorem 8. Let

$$N := \{x \mid x \in I,\ \varphi'(x) = 0\}.$$

Then, by the continuity of φ', N is a closed set, whose interior, $\operatorname{int} N = \emptyset$. Thus $I \cap (\mathbb{R} \setminus N)$ is open, that is, it can be obtained as a union of at most countably infinite disjoint open intervals. Let $J =]a, b[\subset I \cap (\mathbb{R} \setminus N)$ be an interval of *maximal* length for which $a, b \notin I \cap (\mathbb{R} \setminus N)$. According to the definition, if $a \in I$ then $\varphi'(a) = 0$ and the same holds for b. If $a = -\infty$ and $b = \infty$ then according to the definition $J =]-\infty, \infty[= I = \mathbb{R}$ (and $N = \emptyset$). If this is not the case then by the maximality either $J = I$ (and then $N = \emptyset$) or at least one of the endpoints of J belongs to I.

These conditions guarantee the existence of a function $\psi \in CM(I)$ for which the functional equation

$$\varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right)=\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) \tag{6.7}$$

holds for all $x, y \in I$. Then (6.7) also holds for all $x, y \in J$, where obviously $\varphi'(x) \neq 0$ if $x \in J$ and $\varphi, \psi \in CM(J)$. Thus, by Lemma 2, ψ' exists and $\psi'(x) \neq 0$ if $x \in J$. Consequently, equation (6.7) can be differentiated in J with respect to x, that is,

$$\frac{\varphi'(x)-\varphi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\varphi'\left(A_{\varphi}^{*}(x,y)\right)}=\frac{\psi'(x)}{2\psi'\left(A_{\psi}(x,y)\right)}$$

follows for all $x, y \in J$. From this, by the symmetry of A_{φ}^{*} and A_{ψ}, we have

$$\frac{\varphi'(x)-\varphi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\psi'(x)}=\frac{\varphi'(y)-\varphi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\psi'(y)} \tag{6.8}$$

for all $x, y \in J$. We introduce the following notation:

$$\varepsilon_{\chi}:=\begin{cases}1 & \text{if } \chi \text{ is increasing}\\ -1 & \text{if } \chi \text{ is decreasing}\end{cases}$$

for all $\chi \in CM(I)$, and let

$$F(x):=\frac{\varepsilon_{\varphi}}{\varphi'(x)},\quad G(x):=\frac{\varepsilon_{\psi}\varphi'(x)}{\psi'(x)\varepsilon_{\varphi}}\qquad \text{if } x \in J. \tag{6.9}$$

Then from (6.8) we have that the functional equation

$$2F\left(\frac{x+y}{2}\right)(G(x)-G(y))=F(x)G(x)-F(y)G(y) \tag{6.10}$$

holds for all $x, y \in J$, where F is (continuous and) *differentiable* in J because φ is twice differentiable. (6.10) implies

$$G(y)\left(2F\left(\frac{x+y}{2}\right)-F(y)\right)=G(x)\left(2F\left(\frac{x+y}{2}\right)-F(x)\right)$$

for all $x, y \in I$, from which, by the continuity of F, the existence of

$$\lim_{y\to x} G(y) = \lim_{y\to x} G(x)\frac{2F\left(\frac{x+y}{2}\right) - F(x)}{2F\left(\frac{x+y}{2}\right) - F(y)} = G(x)$$

follows, as there exists $\delta > 0$ for which $2F\left(\frac{x+y}{2}\right) - F(y) \neq 0$ if $y \in]x-\delta, x+\delta[\subset J$. Thus G is *continuous* in J. From (6.10) we have

$$\left(2F\left(\frac{x+y}{2}\right) - F(x)\right)\frac{G(x)-G(y)}{x-y} = G(y)\frac{F(x)-F(y)}{x-y}$$

for all $x, y \in J$ with $x \neq y$, from which the reader can easily see (using the continuity of G) that G is *differentiable* in J and if $y \to x$

$$F(x)G'(x) = G(x)F'(x) \quad \text{if } x \in J.$$

This implies $(\log G(x) - \log F(x))' = 0$ $(x \in J)$, that is, there exists $c > 0$ such that $\log G(x) - \log F(x) = \log c$ $(x \in J)$, from which we have

$$G(x) = cF(x) \quad \text{if } x \in J. \tag{6.11}$$

Thus, by (6.11), (6.10) implies (6.4) for all $x, y \in J$, where $F : J \to \mathbb{R}_+$ is a continuous function. By Lemma 3, there exist constants $\alpha, \beta \in \mathbb{R}$ for which

$$F(x) = \alpha x + \beta > 0 \quad \text{if } x \in J. \tag{6.12}$$

This implies, by (6.9)

$$\varphi'(x) = \frac{\varepsilon_\varphi}{\alpha x + \beta} \quad (\alpha x + \beta > 0) \quad \text{if } x \in J. \tag{6.13}$$

According to the definition, the following cases are possible: either $J = I$ or one of the endpoints of J belongs to I; let us denote it by $c \in I$ ($c = a$ or b if $J =]a, b[$). Then $\varphi'(c) = 0$, but this contradicts (6.13) by the continuity of φ' in I. Therefore in any case $N = \emptyset$ and $J = I$. Thus the solution (6.13) is a function defined on the whole of I. Up to equivalence for the generating functions, there are the following possible cases:

(i) $\alpha = 0$ then $\beta > 0$ thus (6.13) implies

$$\varphi(x) = \varepsilon_\varphi \frac{1}{\beta} x + \delta \sim x \quad \text{if } x \in I;$$

(ii) $\alpha > 0$ then (6.13) implies

$$\begin{aligned}\varphi(x) &= \varepsilon_\varphi \frac{1}{\alpha}\log(\alpha x+\beta)+\delta\\ &= \varepsilon_\varphi \frac{1}{\alpha}\log\left(x+\frac{\beta}{\alpha}\right)+\varepsilon_\varphi\frac{1}{\alpha}\log\alpha+\delta \sim \log(x+\lambda)\\ &\text{if } x\in I,\end{aligned}$$

where $\lambda = \frac{\beta}{\alpha} \in P_+$;

(iii) $\alpha < 0$ then

$$\begin{aligned}\varphi(x) &= \varepsilon_\varphi \frac{1}{\alpha}\log(\alpha x+\beta)+\delta\\ &= \varepsilon_\varphi \frac{1}{\alpha}\log\left(-x+\frac{\beta}{-\alpha}\right)+\varepsilon_\varphi\frac{1}{\alpha}\log(-\alpha)+\delta \sim \log(-x+\mu)\\ &\text{if } x\in I,\end{aligned}$$

where $\mu = \frac{\beta}{-\alpha} \in P_-$.

With this we obtain the solutions stated in Theorem 8. It can easily be seen that the A-conjugate means formed with these generating functions are quasi-arithmetic means as well. $\square$

7. Inequalities

If $x, y \in \mathbb{R}_+$ then let

$$H(x,y) := \frac{2xy}{x+y} \tag{7.1}$$

be the well-known harmonic mean. If $I \subset \mathbb{R}$ is an open interval and $\lambda \in P_+$ then let

$$H_\lambda^+(x,y) := H(x+\lambda, y+\lambda) - \lambda \quad (x,y\in I), \tag{7.2}$$

and for $\mu \in P_-$ let

$$H_\mu^-(x,y) := -H(-x+\mu, -y+\mu) + \mu \quad (x,y\in I). \tag{7.3}$$

It can easily be seen that $H_\lambda^+ : I^2 \to I$ ($\lambda \in P_+$) and $H_\mu^- : I^2 \to I$ ($\mu \in P_-$) are means. These means can be formed according to the rules in the following scheme.

If $I =]a, b[$ $(a < b,\ a, b \in \mathbb{R})$ is a *bounded* open interval then $P_+ = \{\lambda \mid \lambda > -a\}$. Then the mean $H_\lambda^+ : I^2 \to I$ can be obtained as follows:

$$\begin{tikzcd}
x < y \ (x, y \in I) \arrow[r, "\lambda \in P_+"] & 0 < x + \lambda < y + \lambda \arrow[d] \\
x < H_\lambda^+(x, y) < y & x + \lambda < H(x + \lambda, y + \lambda) < y + \lambda. \arrow[l]
\end{tikzcd}$$

Similarly we get that $P_- = \{\mu \mid \mu > b\}$ and for $\mu \in P_-$ the mean $H_\mu^- : I^2 \to I$ can be obtained as follows:

$$\begin{tikzcd}
x < y \ (x, y \in I) \arrow[r, "\mu \in P_-"] & -x + \mu > -y + \mu > 0 \arrow[d] \\
& -x + \mu > H(-x + \mu, y + \mu) > -y + \mu \arrow[d, shift right] \\
x < H_\mu^-(x, y) < y & x - \mu < -H(-x + \mu, -y + \mu) < y - \mu. \arrow[l] \arrow[u, shift right]
\end{tikzcd}$$

The cases $I =]-\infty, b[$ $(b \in \mathbb{R})$ and $I =]a, \infty[$ $(a \in \mathbb{R})$ can be handled in a similar way.

Theorem 9. *An A-conjugate mean generated by a twice differentiable function is quasi-arithmetic if and only if it is one of the following:*

(i) $A(x, y)$ $(x, y \in I)$, *or*

(ii) $H_\lambda^+(x, y)$ $(x, y \in I)$ *for some* $\lambda \in P_+$, *or*

(iii) $H_\mu^-(x, y)$ $(x, y \in I)$ *for some* $\mu \in P_-$.

Proof. The statement trivially follows from the form of the generating functions given in Theorem 8. □

Theorem 10. *Let* $x, y \in I$ *and* $x \neq y$. *Then the function* $\lambda \mapsto H_\lambda^+(x, y)$ *is strictly increasing on* P_+, *the function* $\mu \mapsto H_\mu^-(x, y)$ *is strictly decreasing on* P_-, *and the inequality*

$$H_\lambda^+(x, y) < A(x, y) < H_\mu^-(x, y) \quad (\lambda \in P_+,\ \mu \in P_-) \tag{7.4}$$

holds for all $x, y \in I$, $x \neq y$. *Furthermore,*

$$\lim_{\lambda \to \infty} H_\lambda^+(x, y) = \lim_{\mu \to \infty} H_\mu^-(x, y) = A(x, y). \tag{7.5}$$

Proof. Supposing $P_+ \neq \emptyset$ and $P_- \neq \emptyset$ we easily obtain

$$\frac{\partial H_\lambda^+(x,y)}{\partial \lambda} = \frac{(x-y)^2}{(x+y+2\lambda)^2} > 0 \quad \text{and}$$

$$\frac{\partial H_\mu^-(x,y)}{\partial \mu} = \frac{-(x-y)^2}{(-x+-y+2\mu)^2} < 0,$$

from which the statements concerning monotony follow. (7.5) can be calculated directly, and it easily implies (7.4). □

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