

Isometric immersion of complete Riemannian manifolds

By C. L. BEJAN (Iași), T. Q. BINH (Debrecen) and L. TAMÁSSY (Debrecen)

Dedicated to Professor Béla Gyires on his 90th birthday

Abstract. Global isometric mappings φ of a Riemannian manifold M^n into a euclidean space E^{n+m} were investigated under different additional conditions on M^n , φ and m by a number of geometers. We mention here only the works of C. TOMPKINS [7], S. S. CHERN and C. C. HSIUNG [1], S. S. CHERN and N. H. KUIPER [2], H. JACOBOWITZ [3].

Recently S. S. YANG [8] showed that in case of a compact M^n no euclidean ball $B(r) \subset E^{n+m}$ of radius r can contain $\varphi(M^n)$, provided $\|H(\varphi(x))\| < \frac{1}{\sqrt{m}r}$, $\forall x \in M$, where H denotes the mean curvature vector of $\varphi(M^n)$.

In this note we prove a similar theorem, replacing compactness of M^n by the weaker condition of completeness. Our result is somewhat stronger even in the case of compactness, if $m > 1$. Also isometric immersion $\varphi : M^n \rightarrow S^{n+m-1}$ of a complete M^n into a euclidean sphere S^{n+m-1} not pinched by certain geodesic balls of S^{n+m-1} is investigated. Corollary 1 concerns the diameter of $\varphi(M^n)$, and Corollary 2 concerns minimal submanifolds of the sphere.

1. Introduction

Global isometric mappings φ of a Riemannian manifold M into a euclidean space E were investigated under different additional conditions on M , φ and $\dim E$ by a number of geometers. In 1963 S. S. CHERN and C. C. HSIUNG [1] showed that there exists no isometric minimal immersion of a compact Riemannian manifold into a euclidean space. In 1998 S. S. YANG [8] proved the following

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Theorem (YANG [8]). Let $\varphi : M^n \rightarrow E^{n+m}$ be an isometric immersion of a compact Riemannian manifold M^n into the euclidean E^{n+m} , H the mean curvature vector of $\varphi(M^n)$, and $\|H\|$ the euclidean length of H . Then in case of

$$\|H(\varphi(x))\| < \frac{1}{\sqrt{m}r} \quad \forall x \in M$$

no euclidean ball $B(r)$ of radius r can contain $\varphi(M^n)$.

Chern and Hsiung's above mentioned result is a consequence of this theorem. Indeed, since M^n is compact, so is $\varphi(M^n) \subset E^{n+m}$. Hence $\varphi(M^n)$ is bounded, and therefore contained in a sufficiently big $B(R)$. If moreover $\varphi(M^n)$ is a minimal surface, as in Chern and Hsiung's theorem, then $\|H\| \equiv 0$, and thus by Yang's theorem no ball $B(R)$ could contain $\varphi(M^n)$. This is a contradiction, what proves Chern and Hsiung's above cited result.

Yang concluded from the behaviour of $\|H\|$ to the fact that $\varphi(M^n)$ is not pinched into (not contained in) certain balls $B(r)$. However there are results which lead to this conclusion from a restriction on the sectional curvature. H. JACOBOWITZ [3] proved in 1973 that if the sectional curvature of a compact Riemannian manifold M^n is everywhere less than $\frac{1}{r^2}$, then there exists no isometric immersion of M^n into a ball $B(r) \subset E^{2n-1}$. This is a generalization of a result of S. S. CHERN and N. H. KUIPER [2] from 1952. This result ([2; Theorem 5]) says (in an appropriate formulation) that a compact Riemannian manifold M^n with everywhere nonpositive sectional curvature cannot be isometrically embedded in E^{2n-1} . Also this result contains as a corollary the old theorem of C. TOMPKINS [7] from 1939, according to which the flat n -dimensional torus cannot be embedded isometrically in E^{2n-1} .

In this note we want to prove a theorem similar to Yang's theorem, replacing compactness of M^n by the weaker condition of completeness. Also isometric immersion $\varphi : M^n \rightarrow S^{n+m-1}$ of a complete Riemannian M^n into a euclidean sphere not pinched by certain geodesic balls of S^{n+m-1} , will be investigated.

2. Immersion into E^{n+m}

Theorem 1. *Let M^n be a complete Riemannian manifold whose sectional curvature K is bounded away from $-\infty$, and $\varphi : M^n \rightarrow E^{n+m}$ an*

isometric immersion into a euclidean space E^{n+m} , such that $\varphi(M^n)$ is a submanifold of E^{n+m} . Let us denote by H the mean curvature vector of $\varphi(M^n)$, and by $\|H\|$ the euclidean length of H .

Then $\varphi(M^n)$ cannot be contained in any ball $B(r) \subset E^{n+m}$ of radius r satisfying

$$(1) \quad \limsup_{x \in M} \|H(\varphi(x))\| < \frac{1}{r},$$

i.e. $(1) \Rightarrow \varphi(M^n) \not\subset B(r)$.

Both Yang's above result ([8; Theorem 1]) and our Theorem conclude from $\|H\|$ on the upper bound of the radii r of those $B(r)$ which cannot contain $\varphi(M^n)$. However our assumption on M^n is completeness which is weaker than compactness the assumption in Yang's theorem. Moreover, since $\frac{1}{\|H\|} > \frac{1}{\sqrt{m}\|H\|}$ if $m > 1$, our result allows bigger balls not containing $\varphi(M^n)$, than the balls in Yang's theorem. So our theorem deduces from a weaker condition a stronger consequence.

We can achieve this by exploiting a result of H. OMORI ([6] Theorem A.), saying: for any smooth and bounded function ϕ on a complete connected Riemannian manifold Q whose sectional curvatures K are bounded away from $-\infty$, and for any $\varepsilon > 0$ there exists a point $p \in Q$ such that $\|(\text{grad } \phi)(p)\| < \varepsilon$ and at this p the Hessian $\mathcal{H}_\phi$ of ϕ for any unit vector $X \in T_p Q$ is smaller than ε : $\mathcal{H}_\phi(X, X) < \varepsilon$.

PROOF of Theorem 1. Suppose that $\varphi(M^n) \subset B(r)$. We want to show that this assumption contradicts (1). – Without loss of generality we can assume that the center of this ball $B(r)$ is the origin O of the E^{n+m} . Then, denoting the position vector $\overrightarrow{O\varphi(x)}$ $x \in M$ by $F(x)$ and putting $\varphi(M^n) =: Q$,

$$\phi : \varphi(M^n) \rightarrow \mathbb{R}, \quad \phi(x) := \|F(x)\|^2$$

is a smooth function on Q bounded by r^2 .

We want to calculate the Hessian $\mathcal{H}_\phi$ of ϕ . We know that $\mathcal{H}_\phi$ has the form

$$(2) \quad \mathcal{H}_\phi(X, X)\varphi(x) = XX\phi - (\nabla_X X)\phi, \quad X \in \mathfrak{X}(Q),$$

where ∇ denotes the Riemannian connection of Q induced on it by the canonical euclidean connection ∇' of the ambient space E^{n+m} . The relation

$$\nabla'_X Y = \nabla_X Y + \sigma(X, Y), \quad X, Y \in \mathfrak{X}(Q)$$

between ∇ and ∇' , where σ denotes the second fundamental form belonging to the applied embedding, is well known. Applying this for $\mathcal{H}_\phi$ we obtain

$$(3) \quad \mathcal{H}_\phi(X, X) = XX\langle F, F \rangle - (\nabla'_X X)\langle F, F \rangle + \sigma(X, X)\langle F, F \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product. In our further calculation we want to make use of two relations. First

$$(4) \quad \nabla'_X F = X.$$

To see this we use in E^{n+m} Descartes coordinates y^A $A, B = \overline{1, n+m}$. Then $F(x) = \overrightarrow{O\varphi(x)} = y^A \frac{\partial}{\partial y^A}$ and $T_{\varphi(x)}Q \ni X = \frac{\partial y^B}{\partial x^i} \lambda^i(x) \frac{\partial}{\partial y^B} \equiv \xi^B \frac{\partial}{\partial y^B}$, $i = \overline{1, n}$ with arbitrary λ^i . Thus in E^{n+m} we have $\nabla'_X F = XF = \xi^B \frac{\partial y^A}{\partial y^B} \frac{\partial}{\partial y^A} = X$. – The second relation is well known:

$$(5) \quad Z\langle U, V \rangle = \langle \nabla'_Z U, V \rangle + \langle U, \nabla'_Z V \rangle, \quad Z, U, V \in \mathfrak{X}(E^{n+m}).$$

Thus

$$(6) \quad X\langle F, F \rangle \stackrel{(5)}{=} 2\langle \nabla'_X F, F \rangle \stackrel{(4)}{=} 2\langle X, F \rangle$$

and

$$X\langle X, F \rangle \stackrel{(5)}{=} \langle \nabla'_X X, F \rangle + \langle X, \nabla'_X F \rangle,$$

that is

$$(7) \quad \langle \nabla'_X X, F \rangle \stackrel{(4)}{=} X\langle X, F \rangle - \langle X, X \rangle.$$

By using these relations we also get

$$(8) \quad \begin{aligned} (\nabla'_X X)\langle F, F \rangle &\stackrel{(5)}{=} 2\langle \nabla'_{\nabla'_X X} F, F \rangle \stackrel{(4)}{=} 2\langle \nabla'_X X, F \rangle \\ &\stackrel{(7)}{=} 2X\langle X, F \rangle - 2\langle X, X \rangle. \end{aligned}$$

Hence, by (3) (6) and (8)

$$\begin{aligned}\mathcal{H}_\phi(X, X) &= 2X\langle X, F \rangle - 2X\langle X, F \rangle + 2\langle X, X \rangle + 2\langle \sigma(X, X), F \rangle \\ &= 2\langle X, X \rangle + 2\langle \sigma(X, X), F \rangle.\end{aligned}$$

Let $E_i(x)$ be an orthonormal base of $T_{\varphi(x)}Q$. Then

$$\sum_{i=1}^n \mathcal{H}_\phi(E_i, E_i) = 2n \left(1 + \left\langle \frac{1}{n} \sum_{i=1}^n \sigma(E_i, E_i), F \right\rangle \right) = 2n(1 + \langle H, F \rangle).$$

Now, applying Omori's cited theorem, for any $\varepsilon > 0$ there exists a $\varphi(x) = p \in Q$ such that $\mathcal{H}_\phi(X, X) < \varepsilon$ for any unit vector $X \in T_p Q$. Thus $\mathcal{H}_\phi(E_i, E_i) < 2\varepsilon$, $\sum_{i=1}^n \mathcal{H}_\phi(E_i, E_i) < 2n\varepsilon$, and

$$\begin{aligned}1 + \langle H, F \rangle(p) &< \varepsilon \\ \varepsilon - 1 > \langle H, F \rangle &\geq -\|H\| \|F\| \\ \|H\| &> \frac{1 - \varepsilon}{\|F\|}.\end{aligned}$$

According to our assumption $\varphi(M^n) = Q \subset B(r)$; i.e. $\|F\| < r$. So

$$\|H\| > \frac{1 - \varepsilon}{r}, \quad \forall \varepsilon > 0.$$

Hence $\limsup_{x \in M} \|H(\varphi(x))\| < \frac{1}{r}$. However this contradicts (1). Therefore in the case of (1) our assumption $\varphi(M^n) \subset B(r)$ cannot be true. $\square$

We can use our result for an estimate of the diameter d of $\varphi(M^n) \subset E^{n+m}$.

An old result of H. E. JUNG [4] says that: Each subset of E^{n+m} of diameter not greater than d lies in a ball $B(R) \subset E^{n+m}$ of radius

$$(9) \quad R \leq \sqrt{\frac{n+m}{2(n+m)+2}} d.$$

Let now $\varphi(M^n)$ of our Theorem 1 be this subset of E^{n+m} . Then $\varphi(M^n) \subset B(R)$ with an R satisfying (9). On the other hand, if M^n, φ

and r satisfy the conditions of our Theorem 1, then $\varphi(M^n)$ cannot be contained in $B(r) : \varphi(M^n) \not\subset B(r)$. Hence

$$r < R \leq \sqrt{\frac{n+m}{2(n+m)+2}} d,$$

i.e. we obtain

Corollary 1. *If M^n , φ and r satisfy the conditions of our Theorem 1, then the diameter d of $\varphi(M^n)$ is greater than $\sqrt{\frac{2(n+m)+2}{n+m}} r$:*

$$d(\varphi(M^n)) \geq \sqrt{\frac{2(n+m)+2}{n+m}} r.$$

3. Immersion into S^{n+m-1}

We want to prove a similar theorem in the case of immersion into a euclidean unit sphere S^{n+m-1} . We apply the notations of the previous section.

Theorem 2. *Let M^m be a complete Riemannian manifold whose sectional curvature K is bounded away from $-\infty$, and $\varphi : M^n \rightarrow S^{n+m-1}$ an isometric immersion such that $\varphi(M^n)$ is a submanifold of S^{n+m-1} .*

Then $\varphi(M^n)$ cannot be contained in any geodesic ball $\tilde{B}(r)$ of S^{n+m-1} of radius r satisfying

$$(10) \quad \limsup_{x \in M} \|H\varphi(x)\| < \frac{\cos r}{2 \sin \frac{r}{2}}.$$

PROOF. The proof runs the same way as the previous theorem with some minor differences.

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on M^n and $S^{n+m-1}(\subset E^{n+m})$ resp., and ∇' the natural connection on E^{n+m} . Then

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \nabla'_X Y = \tilde{\nabla}_X Y - \langle X, Y \rangle N$$

and

$$(11) \quad \nabla_X Y = \nabla'_X Y + \langle X, Y \rangle N - \sigma(X, Y) \quad X, Y \in \mathfrak{X}(Q),$$

where N is the outward unit normal vector of S^{n+m-1} in E^{n+m} . Then for the Hessian $\mathcal{H}_\phi$ of a smooth function ϕ on Q we have (2), and by (11) we obtain

$$\mathcal{H}_\phi(X, X) = (XX - \nabla'_X X + \langle X, X \rangle N - \sigma(X, X)) \phi.$$

Let A be the position vector of the center of a geodesic ball $\tilde{B}(r)$ on S^{n+m-1} and $F(x)$ again the position vector $\overrightarrow{O\varphi(x)}$. Then $\phi(x) := \|F(x) - A\|^2$ is a smooth and bounded function on $\varphi(M^n) = Q \subset S^{n+m-1}$. A calculation using (3), (6), (8), the facts that $N = F$ on $Q \subset S^{n+m-1}$ and $\nabla'_X A = 0$, yields, similarly as in Section 2, that

$$\mathcal{H}_\phi(X, X) = 2(\langle X, X \rangle - \langle X, X \rangle \langle F, F - A \rangle + \langle \sigma(X, X), F - A \rangle).$$

For a unit vector X this gives

$$\mathcal{H}_\phi(X, X) = 2(1 - \langle F, F - A \rangle + \langle \sigma(X, X), F - A \rangle).$$

Let now again $E_i(p)$ $p = \varphi(x)$ be an orthonormal base of $T_p Q$. Then

$$\begin{aligned} \sum_{i=1}^n \mathcal{H}_\phi(E_i, E_i) &= 2n \left(1 + \left\langle \frac{1}{n} \sum_{i=1}^n \sigma(E_i, E_i), F - A \right\rangle - \langle F, F - A \rangle \right) \\ &= 2n(1 + \langle H, F - A \rangle - \langle F, F - A \rangle). \end{aligned}$$

Since ϕ is smooth and bounded, Q as the isometric immersion of M^n is connected and complete, and its sectional curvature is bounded away from $-\infty$, we may apply Omori's theorem, according to which we have a $\varphi(x) = p \in Q$ for any $\varepsilon > 0$ such that $\mathcal{H}_\phi(E_i, E_i)(p) < 2\varepsilon$. Hence

$$\langle H(p), F - A \rangle + 1 - \langle F, F - A \rangle = \langle H(p), F - A \rangle + 1 - \langle F, F \rangle + \langle F, A \rangle < \varepsilon.$$

Taking into consideration that $\|F\|^2 = 1$, and denoting $\langle F(\varphi(x), A \rangle$ by $\Theta(\varphi(x))$, we obtain

$$\varepsilon - \Theta(p) > \langle H(p), F - A \rangle \geq -\|H(p)\| \cdot \|F - A\| = -\|H(p)\| 2 \sin \frac{\Theta(p)}{2}.$$

Thus to any $\varepsilon > 0$ there exists a $p \in Q$, such that

$$\|H(p)\| > \frac{\cos \Theta(p) - \varepsilon}{2 \sin \frac{\Theta(p)}{2}} \quad p \in Q.$$

Assume now that $Q \subset \tilde{B}(r)$ ($r < \pi$). Then $\Theta(\varphi(x)) \leq r$ and thus $\sin \frac{\Theta(\varphi(x))}{2} \leq \sin \frac{r}{2}$ and $\cos \Theta(\varphi(x)) \geq \cos r$. Hence

$$\|H(p)\| > \frac{\cos r - \varepsilon}{2 \sin \frac{r}{2}},$$

and thus

$$(12) \quad \limsup_{x \in M} \|H(\varphi(x))\| \geq \frac{\cos r}{2 \sin \frac{r}{2}}.$$

However in case of (10), (12) cannot be true, and thus our assumption cannot hold; i.e. in case of (10) there exists no $\tilde{B}(r)$ containing $\varphi(M^n) = Q$. $\square$

Finally we want still to show that an interesting result of S. B. MYERS is a consequence of a corollary of our last theorem.

First we remark that we have confined r in our Theorem 2 by π , however, because of the inequality (9), it must be smaller even than $\frac{\pi}{2}$.

Let now $\varphi(M^n)$ be a complete and *minimal* ($H = 0$) submanifold of S^{n+m} . Then by our Theorem 2 $\varphi(M^n)$ cannot be contained in a ball $\tilde{B}(r)$, $r < \frac{\pi}{2}$. So we obtain

Corollary 2. *No complete minimal submanifold of a sphere can be contained in an open hemisphere if K is bounded away from $-\infty$.*

If M^n is compact and $m = 1$, then this is a result of S. B. MYERS ([5; Theorem 4]).

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C. L. BEJAN
CATEDRA MAT. UNIV. TEHNICA "GH. ASACHI"
BDUL. CAROL NR. 11, IAȘI
ROMANIA

E-mail: bejan@math.tuiasi.ro

T. Q. BINH
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H-4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: binh@math.klte.hu

L. TAMÁSSY
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H-4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: tamassy@math.klte.hu

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