

On the completeness of certain sequences

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Abstract. For $A = \{a_1 \leq \dots a_n \leq \dots\} \subset N$ we define $P(A) = \{\sum \varepsilon_i a_i, \varepsilon_i = 0 \text{ or } 1, \sum \varepsilon_i \text{ is finite}\}$. A is said to be subcomplete if $P(A)$ contains an infinite arithmetic progression.

We prove: for every infinite sequence $\{b_m\}$ of integers and for every Graham's sequence $\{[2^n \alpha]\}$ the sequence $\{b_m [2^n \alpha]\}$ is subcomplete.

1. Introduction

Let $A = \{a_1 \leq a_2 \leq \dots \leq a_n \leq \dots\}$ be a sequence of positive integers. Define

$$P(A) = \left\{ \sum_{i=1}^{\infty} \varepsilon_i a_i : \varepsilon_i \in \{0, 1\}; \sum_{i=1}^{\infty} \varepsilon_i < \infty \right\}.$$

Alternatively, we may consider A to be a multiset, and define $P(A)$ to be the set of all sums of the elements of finite submultisets of A .

A is called complete if every sufficiently large integer belongs to $P(A)$. We call A subcomplete if there is an integer $d \neq 0$ such that $P(A)$ contains an infinite arithmetic progression with difference d . For example the powers of 2 form a complete sequence. Let $1 \leq a_1 < a_2 < \dots < a_k < \dots$ be a sequence of integers. Let $A(n) = \sum_{a_i \leq n} 1$. Clearly we have $|\{\sum_{i=1}^k \varepsilon_i a_i : \varepsilon_i \in \{0, 1\}\}| \leq 2^k$. Thus if $\limsup_{n \rightarrow \infty} (n - 2^{A(n)}) = \infty$ then the sequence

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$A = \{1 \leq a_1 < \dots < a_k < \dots\}$ cannot be complete. This implies that if A is complete then

$$A(n) \geq (1 + o(1)) \log_2 n.$$

Denote the set of natural numbers by $\mathbb{N}$, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathbb{R}^+$ be the set of positive real numbers. Given a set A of positive integers, we denote by $A + A$ the set of pairwise sums, $A + A = \{a + a' : a, a' \in A\}$. Let $dA = \{da : a \in A\}$.

Let $g(n) = A(n)/\log n$. Many authors investigated sequences for which $g(n)$ tends to infinity very “slowly” as n tends to infinity. For example P. ERDŐS conjectured and J. BIRCH proved in 1959 (see [1]) that if $(p, q) = 1$, then the sequence formed by the values $p^k q^m$ is complete, where k and m are natural numbers. Clearly these sets have $\Theta(\log^2 n)$ elements less than n and so $g(n) < C(p, q) \log n$.

A few years later J.W. CASSELS proved in [2] the following theorem:
Assume that

$$\lim_{n \rightarrow \infty} (A(2n) - A(n)) / \log \log n = \infty$$

and for every real θ ; $0 < \theta < 1$,

$$\sum_{k=1}^{\infty} \|a_k \theta\| = \infty.$$

Then A is complete, where $\|x\| = \min_{-\infty < n < \infty} \{ |x - n| \}; n \in \mathbb{N}$.

It is easy to see that the theorem of Cassels contains Birch’s result.

In 1969 R.L. GRAHAM asked [3] (see also [4]): for which pairs of positive reals (α, β) is the sequence $A_{\alpha\beta} := \{[2^n \alpha], [2^m \beta] : n, m \in \mathbb{N}\}$ complete? The first author investigated the completeness of this sequence in [5] and [6]. Clearly, here $A_{\alpha\beta}(n) < c(\alpha, \beta) \log n$ and so $g(n) = o(1)$.

The aim of this note is to investigate an Erdős-Birch-Graham-type sequence A for which the conditions of Cassels’ theorem are not valid.

Definition. Let $B = \{b_1 \leq b_2 \leq \dots\}$ be an infinite sequence of integers. Let

$$A(\alpha, B) := \{b_m [2^n \alpha] : a, m \in \mathbb{N}; \alpha \in \mathbb{R}\}.$$

We prove the following

Theorem. *For every real number $\alpha > 0$ and every infinite sequence B of positive integers, $A(\alpha, B)$ is subcomplete.*

We mention here that if $B(n) = O(\log \log n)$ then we cannot use Cassels' theorem, i.e. our theorem is not covered by Cassels' argument. On the other hand it is easy to construct a sequence B so that the second condition of Cassels' theorem is not valid for the sequence $A(\alpha, B)$ either (see [7]). Indeed, first we are going to count the number of elements of $A(\alpha, B)$ between $2^N, 2^{N+1}$, where N is large and a fixed number. So let us assume that

$$2^N \leq [2^n \alpha] b_k < 2^{N+1}$$

or equivalently

$$(1) \quad \frac{2^N}{[2^n \alpha]} \leq b_k < \frac{2^{N+1}}{[2^n \alpha]}.$$

Since $[x] > x/2$ if $x \geq 2$, for $\alpha \geq 2^{1-n}$ we get

$$(2) \quad 2^{N-n} \alpha^{-1} \leq b_k \leq 2^{N+2-n} \alpha^{-1}.$$

This yields

$$\begin{aligned} |[2^N, 2^{N+1}] \cap A(\alpha, B)| &\leq |\cup_{n \geq 0} \{ (2^{N-n} \alpha^{-1}, 2^{N+2-n} \alpha^{-1}) \cap B \}| \\ &\leq 2B(2^{N+2} \alpha^{-1}) = O(\log \log 2^N). \end{aligned}$$

So we conclude that the first condition of Cassels' theorem is not valid for the sequence $A(\alpha, B)$ with $B(n) = O(\log \log n)$.

2. Lemmas

Let $a_n = [2^n \alpha]$ and let $\alpha = \sum_{n=-k}^{\infty} \varepsilon_n(\alpha) 2^{-n}$ where $\varepsilon_n(\alpha) = 0$ or 1 , and assume that $\varepsilon_n(\alpha) = 0$ holds infinitely often.

We need some lemmas:

Lemma 1. *For every n*

$$a_{n+1} = 2a_n + \varepsilon_{n+1}(\alpha).$$

Indeed

$$2^n \alpha = \varepsilon_{-k}(\alpha) \varepsilon_{-k+1}(\alpha) \dots \varepsilon_n(\alpha) \cdot \varepsilon_{n+1}(\alpha) \dots$$

and

$$2^{n+1} \alpha = \varepsilon_{-k}(\alpha) \varepsilon_{-k+1}(\alpha) \dots \varepsilon_n(\alpha) \varepsilon_{n+1}(\alpha) \cdot \dots$$

(in base 2).

So $a_{n+1} = \varepsilon_{-k}(\alpha) \dots \varepsilon_{n+1}(\alpha) = 2a_n + \varepsilon_{n+1}(\alpha)$ (in base 2).

Lemma 2. *Let $\alpha = \sum_{n=-k}^{\infty} \varepsilon_n(\alpha) 2^{-n}$. Assume that $\varepsilon_n(\alpha) = 1$ holds infinitely often. Let $H = \{h_1 < h_2 < \dots\}$ be an infinite sequence of integers. Then $P(\{a_n : n \in \mathbb{N}\} \cup H)$ contains arbitrarily large intervals.*

This lemma is essentially Theorem 3 in [6].

Lemma 3. *Let $X = \{x_1 < x_2 < \dots\}$ be an infinite sequence of integers. Assume that there exists a positive integer $T \geq x_1$ such that for every n*

$$(3) \quad x_{n+1} \leq x_1 + x_2 + \dots + x_n + T.$$

Let $P(X) = \{y_1 < y_2 < \dots\}$. Then $P(X)$ has bounded gaps, in fact

$$y_{n+1} - y_n \leq T$$

for $n = 1, 2, \dots$.

PROOF of Lemma 3. Clearly, we only have to show that for every n the longest gap of $P(\{x_1, \dots, x_n\})$ in the interval $[1, s_n]$, where $s_n = x_1 + \dots + x_n$, is at most T . We prove it by induction on n . For $n = 1$ this is the condition for x_1 . Let now $n > 1$. By the inductive hypothesis the longest gap in $[1, s_{n-1}]$ is at most T , so we are going to seek the longest gap in $[s_{n-1}, s_n]$. Since $x_n < s_{n-1} + T$, and by

$$P(\{x_1, \dots, x_n\}) = \{0, x_n\} + P(\{x_1, \dots, x_{n-1}\}),$$

the longest gap in $[1, s_n]$ is at most $\max\{T, \text{the longest gap in } [1, s_{n-1}]\} = T$. $\square$

Lemma 3 is proved.

Corollary. *Let $X = \{x_1 < x_2 < \dots\}$ be an infinite sequence of integers. Assume that there exists a positive integer N such that for every $n > N$*

$$x_{n+1} \leq x_1 + \dots + x_n.$$

Then $P(X)$ has bounded gaps.

Lemma 4. *Let $x_1, x_2, \dots, x_r$ be any r integers. Then there is a $y > 0$, $y \in P(\{x_1, \dots, x_r\})$ for which $r|y$.*

Lemma 4 is well-known.

3. Proof of the Theorem

Let us assume that there exists an integer N , such that for $n > N$ $\varepsilon_n(\alpha) = 0$. By Lemma 1 we get

$$\begin{aligned} P(\{b_1 a_N, b_1 \cdot 2a_N, b_1 \cdot 4a_N, \dots\}) &= \left\{ b_1 a_N \sum_{i=1}^{\infty} \delta_i 2^i : \delta_i \in \{0, 1\}, \sum \delta_i < \infty \right\} \\ &= \{b_1 a_N \cdot n : n \in \mathbb{N}\} \subset P(A(\alpha, B)), \end{aligned}$$

so $A(\alpha, B)$ is subcomplete in this case.

Now let us assume $|\{n : \varepsilon(\alpha) = 1\}| = \infty$. Let $r = b_1$ and let us consider the integers $b_2, b_3, \dots, b_{r+1}$. By Lemma 4 we can select integers $b_{i_1}, b_{i_2}, \dots, b_{i_t}$, $1 \leq t \leq r$ for which

$$(4) \quad r | b_{i_1} + b_{i_2} + \dots + b_{i_t}.$$

Let

$$h_n = \frac{b_{i_1} + b_{i_2} + \dots + b_{i_t}}{r} \cdot a_n,$$

$$H = \{h_1 < h_2 < \dots\}.$$

Then by Lemma 2 $P(\{a_n\} \cup H)$ contains arbitrarily large intervals. By (4)

$$(5) \quad P(\{b_1 a_n\} \cup \{(b_{i_1} + b_{i_2} + \dots + b_{i_t}) \cdot a_n\}) = b_1 \cdot P(\{a_n\} \cup H),$$

thus for every K there is an integer z such that

$$(6) \quad \{z, z + b_1, z + 2b_1, \dots, z + Kb_1\} \subset b_1 \cdot P(\{a_n\} \cup H) \subset P(A(\alpha, B)).$$

Now let us consider the elements $b_{r+2}, b_{r+3}, \dots, b_{2r+1}$ and $b_{2r+2}, b_{2r+3}, \dots, b_{3r+1}$. By Lemma 4 we can choose elements

$$\{b_{j_1}, b_{j_2}, \dots, b_{j_s}\} \subseteq \{b_{r+2}, b_{r+3}, \dots, b_{2r+1}\}$$

and

$$\{b_{k_1}, b_{k_2}, \dots, b_{k_y}\} \subseteq \{b_{2r+2}, b_{2r+3}, \dots, b_{3r+1}\}$$

for which

$$(7) \quad r | U := \{b_{j_1} + b_{j_2} + \dots + b_{j_s}\}$$

and

$$(8) \quad r|V := \{b_{k_1} + b_{k_2} + \dots, b_{k_y}\}.$$

Clearly $P(\{Ua_n\} \cup \{Va_m\}) \subset P(A(\alpha, B))$.

Lemma 5. $P(\{Ua_n\} \cup \{Va_m\})$ has bounded gaps.

PROOF of Lemma 5. Let $U' = U/r$, $V' = V/r$. Let $\{U'a_n\} \cup \{V'a_m\} = X = \{x_1 \leq x_2 \leq \dots \leq x_n \leq \dots\}$. We are going to show that $P(X)$ has bounded gaps. This implies that $rP(X)$ has also bounded gaps as we wanted. By Lemma 3 we only have to show that there exists N such that for $n > N$

$$(9) \quad x_{n+1} \leq x_n + x_{n-1} + \dots + x_1.$$

For brevity let $A = V'\alpha \log_2 \frac{U'}{V'} + 2U' + 2$. Since $\lim_{z \rightarrow \infty} 2^z/z = \infty$ we can choose a fixed integer z for which

$$(10) \quad 2^z > \min\{z + A/(U'\alpha), V'/(U'\alpha)\}.$$

Let us define s by

$$(11) \quad a_s \leq \frac{U'}{V'} a_z \leq a_{s+1}$$

and let $x_{n+1} = U'a_z$. By (10) an easy calculation shows that $2^{s-1}\alpha > 1$. By (11), for $n > N$ we have

$$(12) \quad \{U'a_1, U'a_2, \dots, U'a_{z-1}, V'a_1, \dots, V'a_s\} \subset \{x_1, \dots, x_n\}.$$

We claim that

$$(13) \quad U'a_z \leq U'a_1 + \dots + U'a_{z-1} + V'a_1 + \dots + V'a_s,$$

which by (12) implies (9). Suppose the claim is false and so

$$(14) \quad \begin{aligned} U'2^z\alpha &\geq U'a_z \geq U'(a_1 + \dots + a_{z-1}) + V'(a_1 + \dots + a_s) \\ &> U'\alpha(1 + 2 + \dots + 2^{z-1} - z) + V'\alpha(1 + 2 + \dots + 2^s - s) \\ &= U'\alpha(2^z - 1 - z) + V'\alpha(2^{s+1} - 1 - s), \end{aligned}$$

hence by (11) and (14)

$$\begin{aligned}
 (15) \quad U'\alpha(z+1) &\geq V'\alpha(2^{s+1} - s - 1) \\
 &= V'2^{s+1}\alpha - V'\alpha(s+1) \\
 &\geq V'a_{s+1} - V'\alpha(s+1) \\
 &\geq U'a_z - V'\alpha(s+1).
 \end{aligned}$$

We claim

$$(16) \quad (s+1) \leq \log_2 \left(\frac{U'}{V'} \right) + z + 2.$$

Indeed, by (10) and (11) we have

$$2^{s-1}\alpha \leq 2^s\alpha - 1 \leq a_s \leq \frac{U'}{V'}a_z \leq \frac{U'}{V'}2^z\alpha,$$

which implies (16). So by (15)

$$U'a_z - V'\alpha(s+1) \geq U'(2^z\alpha - 1) - V' \left[\log_2 \frac{U'}{V'} + z + 2 \right],$$

which yields $2^z \leq z + A/(U'\alpha)$. This contradicts (10) which proves the lemma.

Let $P(\{Ua_n\} \cup \{Va_m\}) = \{y_1 < y_2 < \dots y_n < \dots\}$. Lemma 5 yields that there is a positive integer L for which

$$(17) \quad y_{n+1} - y_n \leq L; n \in \mathbb{N}$$

and

$$(18) \quad r|y_n.$$

Now by (6) there is an integer z for which

$$(19) \quad \{z, z + b_1, z + 2b_1, \dots, z + Lb_1\} \subset b_1P(\{a_n\} \cup H).$$

Our last step is to find an infinite arithmetic progression in $P(A(\alpha, B))$.

We claim $\{(y_1 + z) + kb_1\}_{k=1}^\infty \subset P(A(\alpha, B))$. Let us consider the element $(y_1 + z) + mb_1$ and let us define n by

$$(20) \quad y_n \leq y_1 + z + mb_1 < y_{n+1}.$$

By (7), (8) and (20) we conclude that

$$m + \frac{y_1 - y_n}{b_1} \in \mathbb{N}$$

and

$$0 \leq m + \frac{y_1 - y_n}{b_1} \leq L.$$

By (19) we have $z + b_1 \left(m + \frac{y_1 - y_n}{b_1} \right) \in b_1 P(\{a_n\} \cup H)$. This implies

$$(y_1 + z) + mb_1 = y_n + z + b_1 \left(m + \frac{y_1 - y_n}{b_1} \right) \in P(A(\alpha, B)),$$

as we wanted. $A(\alpha, B)$ is subcomplete and this establishes our theorem.

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