

On space-time manifolds carrying two exterior concurrent skew symmetric killing vector fields

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Abstract. We analyse the structural properties, from a geometrical point of view, of space-time manifolds carrying two exterior concurrent skew symmetric Killing vector fields.

1. Introduction

Let (M, g) be a general space-time with usual differentiability conditions, and normed hyperbolic metric g . We assume in this paper that M carries two space-like skew-symmetric Killing vector fields X and Y [R], such that:

- a) X and Y are exterior concurrent [2],
- b) both X and Y have as generative the timelike vector field e_4 of an orthonormal vector basis

$$\mathcal{O} = \text{vect}\{e_A \mid A = 1, \dots, 4\}.$$

It is proved that any such manifold M is a space form of curvature -1 and that M is foliated by hypersurfaces M_S tangent to X , Y and e_4 and such that the normal of M_S is a timelike concircular vector field. In addition, the following properties are pointed out:

- (i) If U is any vector field of M and V is any vector field of the exterior concurrent distribution $\{X, Y, e_4\}$, then the Ricci curvature $\mathcal{R}(U, V)$ satisfies:

$$\mathcal{R}(U, V) = 3g(U, V).$$

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- (ii) The square of the length of X and Y define an isoparametric system [11] on M .
- (iii) $(e_4)^b$ is a harmonic form and X^b and Y^b form an eigenspace $E^1(M)$ of eigenvalue 3.

Killing vector fields X (or infinitesimal isometries) play in many aspects a distinguished role in differential geometry [10]. They also play an important role when dealing with manifolds having indefinite metrics [20] (as for instance space-time C^∞ -manifolds).

A vector field X whose covariant differential ∇X (∇ is supposed to be symmetric) satisfies $\nabla X = X \wedge \mathcal{U}$ ($\wedge$: wedge product of vector fields) has been defined [1] as a skew symmetric Killing vector field and $\mathcal{U}$ is called the generator of X . For manifolds M carrying such a vector field X , as for instance solutions to Einstein's equations containing massless fields, like electromagnetic or gravitational waves, this property typically reflects intrinsic features. We will see that in connection with the electromagnetic Faraday 2-form $\mathcal{F}$ when expressed in terms of PFAFFIANS [6] a skew symmetric Killing vector field appears in a quite natural manner.

It is well known that in special relativity, electromagnetism is described in the 4-vector formalism by the Maxwell tensor $(F^{\mu\nu})$, which incorporates both the electric and magnetic field [9]. Equivalently, in the language of forms, there thus exists a 2-form $\mathcal{F}$ on $\mathbb{R}^4$

$$\mathcal{F} = \sum_{\alpha=1}^3 \mathcal{E}_\alpha du^\alpha \wedge du^4 + (\mathcal{B}_1 du^2 \wedge du^3 + \mathcal{B}_2 du^3 \wedge du^1 + \mathcal{B}_3 du^1 \wedge du^2),$$

where u^i ($i = 1, 2, 3, 4$) are coordinates in Minkowski space [13].

Therefore, in general relativity on a space-time manifold M , electromagnetism is introduced by a 2-form on M

$$\mathcal{F} = (\mathcal{E}_a \omega^a) \wedge \omega^4 + \sum_{\text{cycl}} \mathcal{B}_a \omega^b \wedge \omega^c,$$

or, in intrinsic manner

$$\mathcal{F} = -(e_4)^b \wedge \mathcal{E}^b + *(e_4)^b \wedge \mathcal{B}^b,$$

where $\{\omega^A \mid A = 1, \dots, 4\}$ is a local field of orthonormal coframes over M ; $\mathcal{F}$ is called the generalized Faraday 2-form. On the space-time manifold (M, g) [6] ($\mathcal{E}_a$ and $\mathcal{B}_a$ represent the components of the electric and

the magnetic vector fields respectively associated with $\mathcal{F}$; a, b, c are the spacelike indices).

Then if $\mathcal{E}$ and $\mathcal{B}$ coincide with X and Y respectively, one finds that $\mathcal{F}$ is a conformal symplectic form having $3(e_4)^b$ as covector of Lee and $3X^b$ as source form. One also finds that the Poynting covector S^b is expressed by $S^b = *(\mathcal{B}^b \wedge \mathcal{E}^b \wedge (e_4)^b)$ and is an exterior recurrent form [19], having $2(e_4)^b$ as recurrence form.

Finally, some properties of the Lie algebra induced by X, Y and e_4 are pointed out. We are quoting here:

- (i) X, Y and e_4 define a perfect symmetric group;
- (ii) X and Y are affine Killing vector fields;
- (iii) if $X = \mathcal{E}, Y = \mathcal{B}$, then $\mathcal{F}$ is a relative integral invariant of X and an invariant of Y . By interchanging the physical interpretations of X and Y one obtains similar results.

2. Preliminaries

Let (M, g) be a Riemannian or pseudo-Riemannian C^∞ -manifold and let ∇ be the covariant differential operator defined by the metric tensor g (we assume that ∇ is the Levi-Civita connection). Let ΓTM be the set of sections of the tangent bundle, and

$$TM \xrightarrow{b} T^*M \quad \text{and} \quad TM \xleftarrow{\sharp} T^*M$$

the classical isomorphism defined by g (i.e. b is the index lowering operator, and $\sharp$ is the index raising operator).

Following [10], we denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM)$$

the set of vector valued q -forms ($q < \dim M$), and we write for the covariant derivative operator with respect to ∇

$$(1) \quad d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM)$$

(it should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$ unlike $d^2 = d \circ d = 0$). If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the identity

vector valued 1-form and is also called the soldering form of M [7]. Since ∇ is symmetric one has that $d^\nabla(dp) = 0$. A vector field Z which satisfies

$$(2) \quad d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM); \quad \pi \in \Lambda^1 M$$

is defined to be an exterior concurrent vector field [1] (see also [3], [4]). In

(2) π is called the concurrence form and is defined by

$$(3) \quad \pi = \lambda Z^\flat, \quad \lambda \in \Lambda^0 M.$$

In this case, if $\mathcal{R}$ is the Ricci tensor, one has

$$(4) \quad \mathcal{R}(Z, V) = \epsilon(n-1)\lambda g(Z, V)$$

($\epsilon = \pm 1$, $V \in \Xi M$, $n = \dim M$).

If $f \in \Lambda^0 M$, then we set $\text{grad } f = (df)^\sharp$.

Consider the function $F(f_1, \dots, f_q)$. Then if

$$(5) \quad \begin{aligned} \langle (df_i)^\sharp, (df_j)^\sharp \rangle &= A_{ij}(F), \Delta f_i = B_i(F) \quad \text{and} \\ [(df_i)^\sharp, (df_j)^\sharp] &= \sum C_{ij}^k(F) \nabla f_k, \end{aligned}$$

where A_{ij} , B_i , C_{ij}^k are smooth functions, one says following [11], that $f_1, \dots, f_q$ define an isoparametric system.

Let $\mathcal{O} = \{e_A \mid A = 1, \dots, n\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \text{covect}\{\omega^A\}$ be its associated coframe. Then E. Cartan's structure equations written in indexless manner are

$$(6) \quad \nabla e = \theta \otimes e,$$

$$(7) \quad d\omega = -\theta \wedge \omega,$$

$$(8) \quad d\theta = -\theta \wedge \theta + \Theta.$$

In the above equations θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature forms on M).

3. Exterior concurrent Killing vector fields

Let (M, g) be a space-time manifold with metric tensor g and let $\mathcal{O} = \{e_A \mid A = 1, \dots, 4\}$ be a local field of orthonormal frames over M

and $\mathcal{O}^* = \text{covect}\{\omega^A\}$ its associated coframe. We assume that the indices $a, b \in \{1, 2, 3\}$ correspond to the spacelike vector fields of $\mathcal{O}$, whilst e_4 corresponds to the timelike vector field of $\mathcal{O}$. Then according to [12] (see also [4]), the soldering form dp is expressed by:

$$(9) \quad dp = - \sum \omega^a \otimes e_a + \omega^4 \otimes e_4.$$

By reference to [8] and in consequence of (9) one has the following structure equations:

$$(10) \quad \begin{aligned} \nabla e_a &= -\theta_a^b \otimes e_b + \theta_a^4 \otimes e_4, \\ \nabla e_4 &= -\theta_4^a \otimes e_a; \end{aligned}$$

$$(11) \quad \begin{aligned} d\omega^a &= -\omega^b \wedge \theta_b^a + \omega^4 \wedge \theta_4^a, \\ d\omega^4 &= -\omega^a \wedge \theta_a^4, \end{aligned}$$

and

$$(12) \quad \begin{aligned} d\theta_b^a &= \Theta_b^a - \theta_b^c \wedge \theta_c^a + \theta_b^4 \wedge \theta_4^a, \\ d\theta_4^a &= \Theta_4^a - \theta_4^b \wedge \theta_b^a. \end{aligned}$$

Let now

$$(13) \quad X = -X^a e_a, \quad \text{and} \quad Y = -Y^a e_a; \quad X^a, Y^a \in \Lambda^0 M,$$

be two skewsymmetric Killing vector fields (in the sense of [1], see also [4]) and assume in a first step that both X and Y have as generative the unit timelike vector field e_4 . Hence, the covariant differentials of X and Y are expressed by

$$(14) \quad \begin{aligned} \nabla X &= X \wedge e_4 \iff \nabla X = \omega^4 \otimes X - X^b \otimes e_b, \\ \nabla Y &= Y \wedge e_4 \iff \nabla Y = \omega^4 \otimes Y - Y^b \otimes e_b. \end{aligned}$$

In order to simplify, we agree in the following to set: $X^b = \alpha$, $Y^b = \beta$. On the other hand, if Z is any spacelike vector field on M , then making use of the structure equations (10), one finds:

$$(15) \quad \nabla Z = -(dZ^a - Z^b \theta_b^a) \otimes e_a - (Z^a \theta_a^4) \otimes e_4.$$

Then by (14) and (15) one calculates

$$(16) \quad \begin{aligned} dX^a &= X^b \theta_b^a + X^a \omega^4, \\ dY^a &= Y^b \theta_b^a + Y^a \omega^4, \end{aligned}$$

and

$$(17) \quad \begin{aligned} d\alpha &= 2\omega^4 \wedge \alpha, \\ d\beta &= 2\omega^4 \wedge \beta, \end{aligned}$$

which shows that α and β are exterior recurrent forms [19] having $e_4^b = \omega^4$ as recurrence form. We recall [1] that equations (17) are consequences of (14). In addition, it can be seen from (17) that one has

$$(18) \quad d\omega^4 = 0 \Rightarrow \theta_4^a = \lambda \omega^a, \quad \lambda \in \Lambda^0 M.$$

With the help of (18) one quickly finds

$$(19) \quad \nabla e_4 = \lambda(dp - \omega^4 \otimes e_4),$$

which shows that e_4 is torse forming (see also [4]) [14].

In a second step, operating on ∇X and ∇Y by the covariant derivative operator d^∇ , one derives by (17) and (19)

$$(20) \quad \begin{aligned} d^\nabla(\nabla X) &= \nabla^2 X = \lambda \alpha \wedge dp + (\lambda - 1)(\omega^4 \wedge \alpha) \otimes e_4, \\ d^\nabla(\nabla Y) &= \nabla^2 Y = \lambda \beta \wedge dp + (\lambda - 1)(\omega^4 \wedge \beta) \otimes e_4. \end{aligned}$$

Hence, by (2) the necessary and sufficient condition in order that X and Y be exterior concurrent vector fields is that $\lambda = 1$, and in this case equations (20) go over into

$$(21) \quad \begin{aligned} \nabla^2 X &= \alpha \wedge dp, \\ \nabla^2 Y &= \beta \wedge dp. \end{aligned}$$

Since $\alpha = X^b$, $\beta = Y^b$, one may develop equations (21) as:

$$(22) \quad \begin{aligned} X^a(\theta_a^b \otimes e_b - \theta_a^4 \otimes e_4) &= \left(\sum_a X^a \omega^a \right) \wedge (\omega^b \otimes e_b - \omega^4 \otimes e_4), \\ Y^a(\theta_a^b \otimes e_b - \theta_a^4 \otimes e_4) &= \left(\sum_a Y^a \omega^a \right) \wedge (\omega^b \otimes e_b - \omega^4 \otimes e_4), \end{aligned}$$

and one gets

$$(23) \quad \Theta_B^A = -\omega^A \wedge \omega^B.$$

Hence, according to a well known formula, equation (23) shows the fact that the manifold M under consideration is a space form of curvature -1 . In this condition ($\lambda = 1$) one easily finds that e_4 is also an exterior concurrent vector field. Since this property is preserved by linearity, one may say that $\mathcal{D} = \{X, Y, e_4\}$ defines an exterior concurrent distribution on M . Now, one can check that $\mathcal{D}$ is also involutive. It should also be noticed that the existence of the exterior concurrent skew symmetric Killing vector fields X and Y is determined by the closed differential system defined by (17) and (18). Further, it is not hard to show that if $N = \sum N^a e_a$ is a vector field orthogonal to $\mathcal{D} = \{X, Y, e_4\}$, then the differentials of N^a satisfy

$$dN^a = N^b \theta_b^a.$$

Then with the help of the first equations (10) one derives by (15) that

$$(24) \quad \nabla N = -N^b \otimes e_4,$$

which shows that N is a timelike concircular vector field. Therefore, one may say that the manifold M under consideration is foliated by hypersurfaces M_D whose normals are timelike concircular.

Next, taking into account the signature of g , and making use of the general formula

$$\mathcal{R}(U, Z) = \sum \langle R(e_A, U \mid Z, e_A), \quad U, Z \in \Xi M,$$

($\mathcal{R}$ (resp. R) denotes the Ricci tensor field of ∇ (resp. the curvature tensor field)), one derives:

$$(25) \quad \mathcal{R}(U, V) = 3g(U, V),$$

where V is any vector field of the distribution D . On the other hand recall that the covariant derivative $\nabla \omega$ of a 1-form $\omega = \omega_A \omega^A$, $\omega_A \in \Lambda^0 M$ is expressed by

$$(26) \quad \nabla \omega = (d\omega_A - \omega_B \theta_A^B) \otimes \omega^A,$$

(see also [20]) and following [15], ω is a Killing form if

$$(27) \quad \nabla\omega = 0, \quad \delta\omega = 0.$$

Coming back to the case under discussion, one finds by (17) that α and β satisfy conditions (27), which means that they are Killing 1-forms.

Finally, setting

$$2l_x = \|X\|^2, \quad 2l_y = \|Y\|^2,$$

one gets by (14)

$$(28) \quad \begin{aligned} (d2l_x)^\sharp &= 4l_x e_4 \Rightarrow \|(d2l_x)^\sharp\|^2 = 16l_x^2, \\ (d2l_y)^\sharp &= 4l_y e_4 \Rightarrow \|(d2l_y)^\sharp\|^2 = 16l_y^2, \end{aligned}$$

and

$$(29) \quad \begin{aligned} \nabla(d2l_x)^\sharp &= -4l_x \omega^4 \otimes e_4, \\ \nabla(d2l_y)^\sharp &= -4l_y \omega^4 \otimes e_4. \end{aligned}$$

Next, taking into account the signature of g , one derives from (28)

$$(30) \quad \begin{aligned} \operatorname{div}(d2l_x)^\sharp &= -10(2l_x), \\ \operatorname{div}(d2l_y)^\sharp &= -10(2l_y), \end{aligned}$$

and

$$(31) \quad [(d2l_x)^\sharp, (d2l_y)^\sharp] = 0.$$

Hence, by reference to (5), equations (28), (30) and (31) show that $\|X\|^2$ and $\|Y\|^2$ define an isoparametric system on M . Summarizing, we can formulate the following

Theorem 3.1. *Let (M, g) be a space-time manifold carrying two space-like vector fields X and Y which have the property to be exterior concurrent skewsymmetric Killing vector fields. If the generatives of X and Y coincide with the unit timelike vector field e_4 of M , then M is a space form of curvature -1 . In addition, one has the following properties:*

- (i) (M, g) is foliated by hypersurfaces M_D tangent to the exterior concurrent distribution $\mathcal{D} = \{X, Y, e_4\}$ and the normal N of M_D is a timelike concircular vector field, i.e.

$$\nabla N = -N^\flat \otimes e_4;$$

- (ii) if U is any vector field of M and $V \in \mathcal{D}$, then the Ricci curvature $\mathcal{R}(U, V)$ satisfies:

$$\mathcal{R}(U, V) = 3g(U, V);$$

- (iii) $X^\flat$ and $Y^\flat$ are Killing forms;
 (iv) the square of the length of X and Y define an isoparametric system on M .

4. Harmonic properties on M

Let σ be the volume element of the manifold M under discussion, and let $*$ be the star operator determined by a local orientation of M . One has

$$(32) \quad *\omega^4 = \omega^1 \wedge \omega^2 \wedge \omega^3,$$

and taking into account (18) (remember that $\lambda = 1$), one finds by (11)

$$(33) \quad d*\omega^4 = -3\sigma.$$

Next, according to the general formula

$$(34) \quad \delta u = (-1)^{n(p+1)+1} * d * u, \quad u \in \Lambda^p M,$$

one finds with (33) that $\delta\omega^4 = 0$ and therefore by (18) one derives that

$$\Delta\omega^4 = 0,$$

which shows that ω^4 is a harmonic form.

Next, since $\alpha = X^\flat = -\sum_a X^a \omega^a$, one deduces by (16) and with the help of (11) that

$$\delta\alpha = d*\alpha = 0,$$

and taking account of (17) one deduces

$$(35) \quad \Delta\alpha = \delta(2\omega^4 \wedge \alpha) = 3\alpha.$$

Clearly, in a similar way, one has

$$(36) \quad \Delta\beta = 3\beta,$$

and the above equations show that α and β are eigenfunctions of Δ having 3 as eigenvalue (see also [16]). Therefore, one may state that α and β define an eigenspace $E^1(M)$ of eigenvalue 3.

Consequently, we have

Proposition 4.1. *The dual forms $X^\flat, Y^\flat$ of the exterior concurrent skewsymmetric Killing vector fields X and Y of M , define an eigenspace $E^1(M)$ of eigenvalue 3.*

5. Generalized Faraday form

Following [6] the 2-form

$$(37) \quad \mathcal{F} = (\mathcal{E}_a \omega^a) \wedge \omega^4 + \sum_{\text{cycl}} \mathcal{B}_a \omega^b \wedge \omega^c,$$

where $\mathcal{E}_a$ (resp. $\mathcal{B}_a$) are the components of the electric field $\mathcal{E}$ (resp. the components of the magnetic vector field) is called the generalized Faraday form of a space-time manifold M . In order to agree with (13) one may express (37) in an intrinsic manner as

$$\mathcal{F} = -(e_4)^\flat \wedge \mathcal{E}^\flat + *(e_4)^\flat \wedge \mathcal{B}^\flat.$$

It should be noticed that the above expression of $\mathcal{F}$ is in accordance with the expression of $\mathcal{F}$ in case M is a Minkowski manifold [18].

Assume now that $\mathcal{E}(\mathcal{E}_a)$ and $\mathcal{B}(\mathcal{B}_a)$ coincide with the Killing vector field X and the Killing vector field Y , respectively. Then setting

$$\phi = \sum_{\text{cycl}} \mathcal{B}_a \omega^b \wedge \omega^c = \sum_{\text{cycl}} Y^a \omega^b \wedge \omega^c,$$

($\sum_{\text{cycl}}$: cyclic permutation of the spacelike indices a, b, c) one may write

$$(38) \quad \mathcal{F} = X \wedge \omega^4 + \phi.$$

Making use of the equations (11), one gets by the equations (16) that

$$(39) \quad d\phi = 3\omega^4 \wedge \phi,$$

i.e. ϕ is an exterior recurrent 2-form having $3\omega^4$ as recurrence form [19]. We notice that, since $d(\alpha \wedge \omega^4) = 0$, (see 17), one may write

$$d\mathcal{F} = d\phi \iff \mathcal{F} \sim \phi,$$

i.e. $\mathcal{F}$ and ϕ belong to the same class of homology. Therefore one may write

$$(40) \quad d\mathcal{F} = 3\omega^4 \wedge \mathcal{F}, \quad (\text{remember } \omega^4 = (e_4)^\flat),$$

which shows that $\mathcal{F}$ is a conformal symplectic form, having $3\omega^4$ as covector of Lee.

Setting

$$\psi = \sum_{\text{cycl}} X^a \omega^b \wedge \omega^c,$$

one obtains by (34)

$$(41) \quad *\mathcal{F} = -\beta \wedge \omega^4 + \psi \quad (\beta = Y^\flat).$$

Then by a standard calculation and with the help of the equations (16) one gets

$$(42) \quad \delta\mathcal{F} = 3\alpha.$$

Consequently, as a generalization of the concept of source, in case M is a Minkowski manifold, one may consider in the case under discussion 3α as being the source of the considered generalized Faraday form. Clearly by interchanging the role of $X = \mathcal{E}$ and $Y = \mathcal{B}$, the corresponding Faraday form will have as source 3β . Further, as a generalization of the concept of Poynting covector $S^\flat$ associated with $\mathcal{F}$, one may write in an intrinsic manner

$$(43) \quad S^\flat = *(\mathcal{B}^\flat \wedge \mathcal{E}^\flat \wedge (e_4)^\flat).$$

Since in the case under discussion, one has

$$\mathcal{B}^\flat = \beta, \quad \mathcal{E}^\flat = \alpha, \quad (e_4)^\flat = \omega^4,$$

one finds by a standard calculation

$$(44) \quad S^\flat = \sum_{\text{cycl}} (X^a Y^\flat - X^\flat Y^a) \omega^c.$$

Further, it is worth to emphasize that one may write

$$(45) \quad i_X \mathcal{F} = -2l_x \omega^4 + S^\flat.$$

Making in additon use of the relations (16) and with the help of the equations (12), exterior derivation of (44) gives

$$(46) \quad dS^\flat = 2\omega^4 \wedge S^\flat.$$

This shows that the Poynting covector $S^\flat$ is exterior recurrent [19], and has $2\omega^4$ as recurrence form.

Theorem 5.1. *Let $\mathcal{F}$ be the generalized Faraday form, such that the electric vector field $\mathcal{E}$ (respectively the magnetic vector field $\mathcal{B}$) coincides with the Killing vector field X (respectively the Killing vector field Y). Then $\mathcal{F}$ is a conformal symplectic form, having $3\omega^4$ as covector of Lee, and 3α as associated source. Further if $S^\flat$ denotes the Poynting covector associated with $\mathcal{F}$, then $S^\flat$ is exterior recurrent and has $2\omega^4$ as recurrence form. By interchanging the physical interpretation of X and Y one finds similar results.*

6. Lie algebra induced by the Killing vector fields X and Y

As is known there exists an isomorphism between the Lie algebra $\mathcal{G}$ of the Lie group, and the space of Killing vector fields. If $\omega \in \Lambda^1 M$, then if U and V are any vector fields, we have

$$(47) \quad d\omega(V, U) = 2\omega[V, U].$$

Coming back to the case under consideration, one derives from (14) and (19), since $\lambda = 1$

$$(48) \quad [X, Y] = 0, \quad [X, e_4] = 0, \quad [Y, e_4] = 0,$$

which shows that the vector fields X, Y and e_4 define a commutative triple. We notice by (47) that one has $d\omega(X, Y) = 0$, for any $\omega \in \Lambda^1 M$.

Moreover one finds by (17)

$$(49) \quad \mathcal{L}_X \alpha = 0, \quad \mathcal{L}_Y \beta = 0,$$

and since $\alpha = X^\flat$, $\beta = Y^\flat$ one may say that the dual form of X and Y are self-invariant.

Next, taking the Lie derivatives of ∇X and ∇Y , one gets by (14) and (49)

$$(50) \quad \mathcal{L}_X(\nabla X) = 0, \quad \mathcal{L}_Y(\nabla Y) = 0.$$

This proves the fact that X and Y are affine Killing vector fields (see also [10]).

Further, by (45) one calculates

$$(51) \quad \mathcal{L}_X \mathcal{F} = -3\omega^4 \wedge i_X \mathcal{F} = -\omega^4 \wedge S^\flat,$$

which by (46) gives

$$(52) \quad d(\mathcal{L}_X \mathcal{F}) = 0.$$

Hence according to [17] the above equation confirms the fact that $\mathcal{F}$ is a relative integral invariant of X . Moreover, let $\mathbb{L}$ be the (1.1) type operator on forms defined by [5], that is:

$$\mathbb{L}u = u_1 = u \wedge \Omega,$$

where Ω is an almost symplectic form. In the case under discussion we set

$$\alpha_1 = \alpha \wedge \mathcal{F},$$

and by (49) we deduce by exterior differentiation

$$d(\mathcal{L}_X \alpha_1) = 0.$$

Therefore, we may assert that the property of integral invariance of X is preserved by the operator $\mathbb{L}$.

Finally, setting $s = g(X, Y)$, one quickly derives by (14)

$$(53) \quad ds = 2s\omega^4.$$

Next, since by (38) and (51) we deduce

$$(54) \quad i_Y \mathcal{F} = s\omega^4 \Rightarrow d(i_Y \mathcal{F}) = 0,$$

it quickly follows by (40)

$$(55) \quad \mathcal{L}_Y \mathcal{F} = 0.$$

This proves the fact that the generalized Faraday form having X as associated electric vector field is invariant by the magnetic vector field Y . Moreover, making use of the operator $\mathbb{L}$, i.e.

$$\mathbb{L}\beta = \beta_1 = \beta \wedge \mathcal{F},$$

then one calculates that

$$\mathcal{L}_Y \beta_1 = 0.$$

Hence the property of $\mathcal{F}$ to be invariant under Y is preserved by the operator $\mathbb{L}$. Clearly, the above properties hold by interchanging the role of the Killing vector fields X and Y .

We state the

Theorem 6.1. *Let (M, g) be the space-time manifold carrying two spacelike Killing vector fields X and Y defined in Section 3. Then regarding the Lie algebra induced by X and Y , we have the following properties:*

(i) $X^\flat$ and $Y^\flat$ are selfinvariant, i.e.

$$\mathcal{L}_X X^\flat = 0, \quad \mathcal{L}_Y Y^\flat = 0;$$

(ii) X and Y are affine Killing vector fields, i.e.

$$\mathcal{L}_X(\nabla X) = 0, \quad \mathcal{L}_Y(\nabla Y) = 0;$$

(iii) let $\mathcal{F}$ be the generalized Faraday form on M , and assume that X and Y coincide with the electric vector field $\mathcal{E}$ and the magnetic vector field $\mathcal{B}$ respectively, associated with $\mathcal{F}$.

Then the following properties hold:

(a) $\mathcal{F}$ is a relative integral invariant of X and this property is preserved by Weyl's operator $\mathbb{L}$, i.e.

$$d(\mathcal{L}_X \mathcal{F}) = 0, \quad d(\mathcal{L}_X(\mathbb{L}X^\flat)) = 0, \quad (\mathbb{L}X^\flat = X^\flat \wedge \mathcal{F});$$

(b) $\mathcal{F}$ is invariant by Y and the same property is preserved by $\mathbb{L}$, i.e.

$$\mathcal{L}_Y \mathcal{F} = 0, \quad \mathcal{L}_Y(\mathbb{L}Y^\flat) = 0, \quad (\mathbb{L}Y^\flat = Y^\flat \wedge \mathcal{F}).$$

By interchanging the physical interpretations of X and Y one obtains similar properties.

References

- [1] R. ROSCA, On exterior concurrent skew symmetric Killing vector fields, *Rend. Mat. Messina* **2** (1993), 131–145.
- [2] R. ROSCA, Exterior concurrent vectorfields on a conformal cosymplectic manifold admitting a Sasakian structure, *Libertas Math. (Univ. Arlington, Texas)* **6** (1986), 167–174.
- [3] M. PETROVIC, R. ROSCA and L. VERSTRAELEN, Exterior concurrent vector fields on Riemannian manifolds, *Soochow J. Math.* **15** (1989), 179–187.
- [4] I. MIHAI, R. ROSCA and L. VERSTRAELEN, Some aspects of the differential geometry of vector fields, *Padge, K.U. Brussel* **2** (1996).
- [5] A. WEYL, Sur la theorie des formes différentielles attachées a une variété analytique complexe, *Comm. Math. Helv.* **20** (1947), 110–116.
- [6] F. DEFEVER and R. ROSCA, On space-time manifolds carrying a skew symmetric Killing vector field, *J. of Geometry and Physics*, (in print).
- [7] J. DIEUDONNÉ, Treatise on Analysis Vol. 4, *Academic Press, New York*, 1974.
- [8] E. CARTAN, Oeuvres, *Gauthier-Villars, Paris*, 1955.
- [9] J. D. JACKSON, Classical Electrodynamics, 2nd edn, *John Wiley, New York*, 1975.
- [10] N. A. POOR, Differential Geometric Structures, *McGraw Hill, New York*, 1981.
- [11] A. WEST, Isoparametric systems on symmetric spaces, *Geometry and Topology of Submanifolds* **5** (1993), 281–287.
- [12] R. ROSCA and S. IANUS, Space-times carrying a quasirecurrent pairing of vector fields, *Gen. Rel. and Grav.* **8** (1977), 411–420.
- [13] R. K. SACHS and H. WU, General relativity for mathematicians, Graduate Texts in Mathematics 48, *Springer-Verlag, New York*, 1977.
- [14] K. YANO, On torse forming direction in Riemannian spaces, *Proc. Imp. Acad. Tokyo* **20** (1944), 340–345.
- [15] K. YANO, Integral formulas in Riemannian geometry, *M. Dekker, New York*, 1970.
- [16] F. W. WARNER, Foundations of Differentiable Manifolds and Lie Groups, *Scott, Foresman and Co, London*, 1971.
- [17] R. ABRAHAM and J. E. MARSDEN, Foundations of Mechanics, *Addison-Wesley, Reading Mass.*, 1978.
- [18] R. ABRAHAM, J.E. MARSDEN and T. RATIU, Manifolds, Tensor Analysis and Applications, *Springer-Verlag, New York*, 1988.
- [19] D. K. DATTA, Exterior recurrent forms on manifolds, *Tensor NS* **36** (1982), 115–120.
- [20] N. CHOQUET-BRUHAT, C. DEWITT-MORETTE and M. DILLARD-BLEICK, Analysis, Manifolds and Physics, *North-Holland, Amsterdam*, 1982.

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