

L_1 -convergence and strong summability of Hankel transforms

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Abstract. In this paper we analyse L_1 -convergence and strong summability of Hankel transforms.

1. Introduction

As usual, we define the Hankel transform $h_\mu f$ of a measurable function f on $(0, \infty)$ by

$$h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(x) x^{2\mu+1} dx,$$

where J_μ denotes the Bessel function of the first kind and order μ . We assume here that $\mu > -1/2$.

In this paper we study pointwise convergence and strong summability of Hankel transforms.

We consider, for every $1 \leq p \leq \infty$, the space $L_{p,\mu}$ that consists of all those complex valued and measurable functions f on $(0, \infty)$ such that

$$\|f\|_{p,\mu} = \left\{ \int_0^\infty |f(x)|^p x^{2\mu+1} dx \right\}^{1/p} < \infty, \quad \text{when } 1 \leq p < \infty,$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}_{x \in (0, \infty)} |f(x)| < \infty, \quad \text{when } p = \infty.$$

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$L_{p,\mu}$ is endowed with the topology associated to $\|\cdot\|_{p,\mu}$, $1 \leq p \leq \infty$.

C.S. HERZ [12] established that the Hankel transform h_μ defines a bounded operator from $L_{p,\mu}$ into $L_{p',\mu}$, provided that $1 \leq p \leq 2$. Here p' denotes the conjugate of p , that is, $p' = \frac{p}{p-1}$.

The partial Hankel integral $S_T(f, \mu; \cdot)$ is defined by

$$S_T(f, \mu; x) = \int_0^T (xy)^{-\mu} J_\mu(xy) y^{2\mu+1} h_\mu(f)(y) dy, \quad x, T \in (0, \infty),$$

for every $f \in L_{p,\mu}$, $1 \leq p \leq 2$. The definition of the operator S_T can be extended to $L_{p,\mu}$, when $2 < p < \frac{4(\mu+1)}{2\mu+1}$, by using the Hankel convolution (see [4]). Numerous authors (see [6], [14] and [15], amongst others) have investigated the pointwise convergence of $S_T(f, \mu; x)$ to $f(x)$, as $T \rightarrow \infty$. In [1] and [2] we give necessary and sufficient conditions in order that

$$(1) \quad \lim_{T \rightarrow \infty} S_T(f, \mu; x) = f(x), \quad \text{a.e. } x \in (0, \infty).$$

In particular, we prove that if $x^{-\mu-1/2}f$ and $x^{-\mu-1/2}h_\mu(f)$ are in $L_{1,\mu}$ then (1) holds ([2, Theorem 3.1]).

The first objective on this paper is to recover f from $h_\mu(f)$ by means of L_1 -convergence. As Corollary 2.1 shows, in general $S_T(f, \mu; \cdot) \notin L_{1,\mu}$, $T \in (0, \infty)$, when $f \in L_{1,\mu}$. Hence it makes no sense to think about the convergence of $S_T(f, \mu; \cdot)$, as $T \rightarrow \infty$, in $L_{1,\mu}$. In Section 2 we obtain necessary and sufficient conditions on f in order that the following

$$\lim_{T \rightarrow \infty} R_T(f, \mu; \cdot) = f$$

holds, when the limit is understood in $L_{1,\mu}$ and where

$$R_T(f, \mu; x) = - \int_0^T (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+2} \frac{d}{dy} (h_\mu(f)(y)) dy, \\ T, x \in (0, \infty).$$

Note that, according to [19, §5.1 (7)], if $f \in L_{1,\mu} \cap L_{1,\mu+1}$ then $R_T(f, \mu; x) = S_T(f, \mu+1; x)$, $x, T \in (0, \infty)$.

I.I. HIRSCHMAN [13], D.T. HAIMO [11] and F.M. CHOLEWINSKI [5] investigated the convolution operation for the Hankel transformation. Let

f, g be measurable functions on $(0, \infty)$. The Hankel convolution $f \# g$ of f and g is defined by

$$(f \# g)(x) = \int_0^\infty f(y)(\tau_x g)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy, \quad \text{a.e. } x \in (0, \infty),$$

where the Hankel translation τ_x , $x \in (0, \infty)$, is defined through

$$(\tau_x g)(y) = \int_0^\infty g(z) D_\mu(x, y, z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dz, \quad \text{a.e. } y \in (0, \infty),$$

and being

$$D(x, y, z) = \frac{2^{3\mu-1} \Gamma(\mu+1)^2}{\Gamma(\mu+1/2) \sqrt{\pi}} (xyz)^{-2\mu} \Delta(x, y, z)^{2\mu-1}, \quad x, y, z \in (0, \infty),$$

where $\Delta(x, y, z)$ represents the area of the triangle having sides with lengths x, y and z , when such a triangle exists, and $\Delta(x, y, z) = 0$, otherwise.

We established in [4] that the definition of the partial Hankel integral S_T can be written through the Hankel convolution as follows

$$S_T(f, \mu; \cdot) = f \# \varphi_T,$$

for every $T \in (0, \infty)$, $1 \leq p \leq 2$ and $f \in L_{p, \mu}$, where

$$\varphi_T(x) = T^{2\mu+2} (xT)^{-\mu-1} J_{\mu+1}(xT), \quad x, T \in (0, \infty).$$

Motivated by the paper of D.V. GIANG and F. MÓRICZ [8] in Section 3 we analyse the strong summability of the Hankel transforms.

Let $q > 0$. We say that the Hankel transform of $f \in L_{1, \mu}$ is strongly summable of exponent q in $x \in (0, \infty)$ when

$$(2) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |S_\nu(f, \mu; x) - f(x)|^q d\nu = 0.$$

By using Hölder's inequality we conclude that if (2) holds then it also holds when q is replaced by $r \in (0, q)$. Moreover, if $x \in (0, \infty)$ and $S_T(f, \mu; x) \rightarrow f(x)$, as $T \rightarrow \infty$, then (2) holds for every $q > 0$. Hence strong summability is weaker than pointwise convergence.

Throughout this paper C will always denote a positive constant not necessarily the same in each occurrence.

2. L_1 -convergence of Hankel transforms

In this section, inspired in the paper of D.V. GIANG and F. MÓRICZ [10], we give conditions that allow to recover a function $f \in L_{1,\mu}$ from $h_\mu(f)$ by means of convergence in the space $L_{1,\mu}$.

Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous in $(0, \infty)$. According to [19, §5.1 (6)] a partial integration leads to

$$\begin{aligned} S_T(f, \mu; x) &= \int_0^T (xy)^{-\mu} J_\mu(xy) y^{2\mu+1} h_\mu(f)(y) dy \\ &= x^{-2\mu-2} \left\{ (xy)^{\mu+1} J_{\mu+1}(xy) h_\mu(f)(y) \right\}_0^T \\ &\quad - \int_0^T (xy)^{\mu+1} J_{\mu+1}(xy) \frac{d}{dy} (h_\mu(f)(y)) dy \end{aligned}$$

for every $x, T \in (0, \infty)$. Since $f \in L_{1,\mu}$, $h_\mu(f)$ is a bounded function on $(0, \infty)$. Hence, since $z^{-\nu} J_\nu(z)$ is bounded on $(0, \infty)$, we have

$$\lim_{y \rightarrow 0^+} (xy)^{\mu+1} J_{\mu+1}(xy) h_\mu(f)(y) = 0, \quad x \in (0, \infty).$$

Then, it follows that

$$(3) \quad S_T(f, \mu; x) = x^{-2\mu-2} (xT)^{\mu+1} J_{\mu+1}(xT) h_\mu(f)(T) + R_T(f, \mu; x),$$

$$x, T \in (0, \infty),$$

where $R_T(f, \mu; x) = - \int_0^T (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+2} \frac{d}{dy} (h_\mu(f)(y)) dy$, $x, T \in (0, \infty)$. Note that if $f \in L_{1,\mu} \cap L_{1,\mu+1}$ then $h_\mu(f)$ is absolutely continuous on $(0, \infty)$ and according to [19, §5.1 (7)], $R_T(f, \mu; x) = S_T(f, \mu+1; x)$, $x, T \in (0, \infty)$.

We will obtain necessary and sufficient conditions in order that the following

$$\lim_{T \rightarrow \infty} R_T(f, \mu; \cdot) = f,$$

holds, in the sense of convergence in $L_{1,\mu}$.

Previously we need to establish some results.

Lemma 2.1. Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous on $(0, \infty)$. Then

$$\begin{aligned} R_T(f, \mu; x) - \sigma_T(f, \mu; x) &= \frac{\lambda^2}{\lambda^2 - 1} (\sigma_{\lambda T}(f, \mu; x) - \sigma_T(f, \mu; x)) \\ &- \frac{2}{(\lambda^2 - 1)T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} h_\mu(f)(y) dy + \tau_T(f, \mu, \lambda; x), \\ x, T &\in (0, \infty) \text{ and } \lambda > 1, \end{aligned}$$

where $\sigma_T(f, \mu; \cdot)$, $T \in (0, \infty)$, denotes the Bochner–Riesz mean of f , that is,

$$\begin{aligned} \sigma_T(f, \mu; x) &= \int_0^T (xy)^{-\mu} J_\mu(xy) \left(1 - \left(\frac{y}{T}\right)^2\right) h_\mu(f)(y) y^{2\mu+1} dy, \\ x, T &\in (0, \infty), \end{aligned}$$

and

$$\begin{aligned} \tau_T(f, \mu, \lambda; x) &= \frac{\lambda^2}{\lambda^2 - 1} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) \left(1 - \left(\frac{y}{\lambda T}\right)^2\right) y^{2\mu+2} \\ &\times \frac{d}{dy} (h_\mu(f)(y)) dy, \end{aligned}$$

for every $x, T \in (0, \infty)$ and $\lambda > 1$.

PROOF. Let $x, T \in (0, \infty)$ and $\lambda > 1$. According to [2, Lemma 2.2] and by (3) we can write

$$\begin{aligned} R_T(f, \mu; x) - \sigma_T(f, \mu; x) &= S_T(f, \mu; x) - \sigma_T(f, \mu; x) \\ &- x^{-2\mu-2} (xT)^{\mu+1} J_{\mu+1}(xT) h_\mu(f)(T) \\ &= \frac{\lambda^2}{\lambda^2 - 1} [\sigma_{\lambda T}(f, \mu; x) - \sigma_T(f, \mu; x)] \\ &- \frac{\lambda^2}{\lambda^2 - 1} \int_T^{\lambda T} y^{2\mu+1} (xy)^{-\mu} J_\mu(xy) \left(1 - \left(\frac{y}{\lambda T}\right)^2\right) h_\mu(f)(y) dy \\ &- x^{-2\mu-2} (xT)^{\mu+1} J_{\mu+1}(xT) h_\mu(f)(T). \end{aligned}$$

Now a partial integration in the last integral allows to conclude the desired equality. $\square$

The following lemma is analogous to the one presented in [9, Lemma 2].

Lemma 2.2. *Let $p \in (3/2, 2]$ and $-1/2 < \mu < p - 2$. Assume that $f \in L_{p,\mu}$. Then*

$$\begin{aligned} & \int_0^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y) dy \right| x^{2\mu+1} dx \\ & \leq Cb(\lambda) \left\{ \frac{1}{T^{2\mu+2}(\lambda^{2\mu+2} - 1)} \int_T^{\lambda T} |f(y)|^p y^{2\mu+1} dy \right\}^{1/p}, \end{aligned}$$

$$T \in (0, \infty) \text{ and } \lambda > 1,$$

where b is a continuous function on $(1, \infty)$ and $b(\lambda) \rightarrow 0$, as $\lambda \rightarrow 1^+$. Here the constant C only depends on p and μ .

PROOF. Let $T \in (0, \infty)$ and $\lambda > 1$. Firstly we split the integral in the left side of (4) as follows

$$\int_0^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y) dy \right| x^{2\mu+1} dx = I_1 + I_2,$$

where

$$I_1 = \int_0^{1/T} \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y) dy \right| x^{2\mu+1} dx$$

and

$$I_2 = \int_{1/T}^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y) dy \right| x^{2\mu+1} dx.$$

We now analyse I_1 . By taking into account that $z^{-\eta} J_\eta(z)$, $\eta \geq -1/2$, is bounded on $(0, \infty)$ we can write that

$$\begin{aligned} I_1 & \leq C\lambda^2 \int_0^{1/T} x^{2\mu+1} \int_T^{\lambda T} |f(y)| y^{2\mu+1} dy dx = C \frac{\lambda^2}{T^{2\mu+2}} \int_T^{\lambda T} |f(y)| y^{2\mu+1} dy \\ & \leq C\lambda^2 (\lambda^{2\mu+2} - 1)^{1/p'} \left\{ \frac{1}{T^{2\mu+2}} \int_T^{\lambda T} |f(y)|^p y^{2\mu+1} dy \right\}^{1/p}. \end{aligned}$$

On the other hand, by virtue of Hausdorff–Young’s inequality for the Hankel transform ([12, Theorem 3]), we have

$$\begin{aligned}
 I_2 &= \frac{1}{T^2} \int_{1/T}^{\infty} \left| \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y) dy \right| x^{-2} x^{2\mu+3} dx \\
 &\leq \frac{1}{T^2} \left\{ \int_{1/T}^{\infty} x^{2\mu+3-2p} dx \right\}^{1/p} \\
 &\quad \times \left\{ \int_{1/T}^{\infty} x^{2\mu+3} \left| \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y) dy \right|^{p'} dx \right\}^{1/p'} \\
 &\leq CT^{-2(\mu+2)/p} \left\{ \int_T^{\lambda T} |f(y)|^p y^{2\mu+3} dy \right\}^{1/p} \\
 &\leq C\lambda^{2/p} \left\{ \frac{1}{T^{2\mu+2}} \int_T^{\lambda T} |f(y)|^p y^{2\mu+1} dy \right\}^{1/p},
 \end{aligned}$$

provided that $\mu < p - 2$.

Hence, we conclude that

$$\begin{aligned}
 &\int_0^{\infty} \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y) dy \right| x^{2\mu+1} dx \\
 &\leq Cb(\lambda) \left\{ \frac{1}{T^{2\mu+2}(\lambda^{2\mu+2} - 1)} \int_T^{\lambda T} |f(y)|^p y^{2\mu+1} dy \right\}^{1/p},
 \end{aligned}$$

where $b(\lambda) = \lambda^2(\lambda^{2\mu+2} - 1) + \lambda^{2/p}(\lambda^{2\mu+2} - 1)^{1/p}$. □

We now characterize the convergence of $R_T(f, \mu; \cdot)$ to f , as $T \rightarrow \infty$, in $L_{1,\mu}$.

Proposition 2.1. *Let $-1/2 < \mu < 0$. Assume that $f \in L_{1,\mu}$, $x^{-\mu-1/2}f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous on $(0, \infty)$. Then*

$$R_T(f, \mu; \cdot) \longrightarrow f, \quad \text{as } T \rightarrow \infty,$$

in $L_{1,\mu}$ if and only if

$$(5) \quad \lim_{\lambda \rightarrow 1^+} \limsup_{T \rightarrow \infty} \int_0^{\infty} |\tau_T(f, \mu, \lambda; x)| x^{2\mu+1} dx = 0.$$

Here $\tau_T(f, \mu, \lambda; x)$ is defined as in Lemma 2.1.

PROOF. According to Lemma 2.1 we can write

$$\begin{aligned} & \left| \int_0^\infty |R_T(f, \mu; x) - \sigma_T(f, \mu; x)| x^{2\mu+1} dx - \int_0^\infty |\tau_T(f, \mu, \lambda; x)| x^{2\mu+1} dx \right| \\ & \leq \frac{\lambda^2}{\lambda^2 - 1} \int_0^\infty |\sigma_{\lambda T}(f, \mu; x) - \sigma_T(f, \mu; x)| x^{2\mu+1} dx \\ & + \frac{2}{(\lambda^2 - 1)T^2} \int_0^\infty \left| \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} h_\mu(f)(y) dy \right| x^{2\mu+1} dx, \end{aligned}$$

$$T \in (0, \infty) \text{ and } \lambda > 1.$$

It is well-known that $\sigma_T(f, \mu; \cdot) \rightarrow f$, as $T \rightarrow \infty$, in $L_{1,\mu}$, provided that $\mu \in (-1/2, 1/2)$ ([7, p. 38]). Hence, for every $\lambda > 1$,

$$\int_0^\infty |\sigma_{\lambda T}(f, \mu; x) - \sigma_T(f, \mu; x)| x^{2\mu+1} dx \rightarrow 0, \quad \text{as } T \rightarrow \infty.$$

Moreover, since $f \in L_{1,\mu}$, $h_\mu(f)$ is bounded on $(0, \infty)$ and then $h_\mu(f) \in L_{p,\mu}(a, b)$, for every $0 < a < b < \infty$ and $1 \leq p < \infty$. Hence, from Lemma 2.2 we deduce, for $-1/2 < \mu < 0$,

$$\begin{aligned} & \int_0^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} h_\mu(f)(y) dy \right| x^{2\mu+1} dx \\ (6) \quad & \leq Cb(\lambda) \left\{ \frac{1}{T^{2\mu+2}(\lambda^{2\mu+2} - 1)} \int_T^{\lambda T} |h_\mu(f)(y)|^2 y^{2\mu+1} dy \right\}^{1/2} \\ & \leq Cb(\lambda) \sup_{y \geq T} |h_\mu(f)(y)|, \quad T \in (0, \infty) \text{ and } \lambda > 1. \end{aligned}$$

Here b is continuous on $(1, \infty)$ and $b(\lambda) \rightarrow 0$, as $\lambda \rightarrow 1^+$ (see Lemma 2.2).

According now to Riemann–Lebesgue Lemma for Hankel transform [18, p. 457] from (6) we infer that

$$\int_0^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} h_\mu(f)(y) dy \right| x^{2\mu+1} dx \rightarrow 0, \quad \text{as } T \rightarrow \infty,$$

uniformly when $1 < \lambda \leq a$, for every $a > 1$.

Thus we conclude that

$$(7) \quad \lim_{T \rightarrow \infty} \int_0^\infty |R_T(f, \mu; x) - \sigma_T(f, \mu; x)| x^{2\mu+1} dx = 0$$

if and only if

$$\lim_{\lambda \rightarrow 1} \limsup_{T \rightarrow \infty} \int_0^\infty |\tau_T(f, \mu, \lambda; x)| x^{2\mu+1} dx = 0.$$

To finish the proof it is sufficient to take into account that, since $\sigma_T(f, \mu; \cdot) \rightarrow f$, as $T \rightarrow \infty$, in $L_{1,\mu}$, (7) is equivalent to $R_T(f, \mu; \cdot) \rightarrow f$, as $T \rightarrow \infty$, in $L_{1,\mu}$. $\square$

A consequence of Proposition 2.1 is the following one.

Corollary 2.1. *Let $-1/2 < \mu < 0$. Assume that $f \in L_{1,\mu}$, $x^{-\mu-1/2}f \in L_{1,\mu}$, that $h_\mu(f)$ is absolutely continuous on $(0, \infty)$ and that (5) holds. If $T \in (0, \infty)$ and $h_\mu(f)(T) \neq 0$, then $S_T(f, \mu; \cdot) \notin L_{1,\mu}$.*

PROOF. It is well-known that [18, p. 199]

$$\sqrt{t}J_\mu(t) = \cos(t + \alpha) + O\left(\frac{1}{t}\right), \quad \text{as } t \rightarrow \infty,$$

for a certain $\alpha \in \mathbb{R}$. Hence, $t^{-\mu-1}J_{\mu+1}(t) \notin L_{1,\mu}$, when $-1/2 < \mu < 0$. By (3) the result follows. $\square$

According to Proposition 2.1 we can find sufficient conditions in order that $R_T(f, \mu; \cdot) \rightarrow f$, as $T \rightarrow \infty$, in $L_{1,\mu}$.

Proposition 2.2. *Let $-1/2 < \mu < p - 2$, with $p \in (3/2, 2]$. Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous on $(0, \infty)$. If we have*

$$(8) \quad \lim_{\lambda \rightarrow 1^+} \limsup_{T \rightarrow \infty} \int_T^{\lambda T} y^{p-1} \left| \frac{d}{dy} h_\mu(f)(y) \right|^p dy < \infty,$$

then (5) is satisfied.

PROOF. Choose $r \in (3/2, p)$ such that $-1/2 < \mu < r - 2$. From Lemma 2.2 it infers that

$$\begin{aligned} & \int_0^\infty |\tau_T(f, \mu, \lambda; x)| x^{2\mu+1} dx \leq C \frac{\lambda^2}{\lambda^2 - 1} T^2 b(\lambda) \\ & \times \left\{ \frac{1}{T^{2\mu+2}(\lambda^{2\mu+2} - 1)} \int_T^{\lambda T} \left(1 - \left(\frac{y}{\lambda T} \right)^2 \right)^r \left| \left(\frac{1}{y} \frac{d}{dy} \right) h_\mu(f)(y) \right|^r y^{2\mu+1} dy \right\}^{\frac{1}{r}}, \\ & T \in (0, \infty) \text{ and } \lambda > 1, \end{aligned}$$

where $b(\lambda) = \lambda^2(\lambda^{2\mu+2} - 1) + \lambda^{2/r}(\lambda^{2\mu+2} - 1)^{1/r}$, $\lambda > 1$.

Hence we conclude after straightforward manipulations that, for every $T \in (0, \infty)$ and $\lambda > 1$,

$$\begin{aligned} & \int_0^\infty |\tau_T(f, \mu, \lambda; x)| x^{2\mu+1} dx \\ (9) \quad & \leq C \frac{b(\lambda)}{(\lambda^{2\mu+2} - 1)^{1/r}} \left\{ \int_T^{\lambda T} y^{r-1} \left| \frac{d}{dy} h_\mu(f)(y) \right|^r dy \right\}^{1/r}. \end{aligned}$$

By using Hölder's inequality it is not hard to see that (8) and (9) imply (5) holds. $\square$

An immediate consequence of Proposition 2.2 is the following.

Corollary 2.2. *Let $-1/2 < \mu < 0$. Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous on $(0, \infty)$. If $y \frac{d}{dy} h_\mu(f)(y) = O(1)$, as $y \rightarrow \infty$, then (5) holds.*

3. Strong summability of Hankel transforms

In this section we study the strong summability of Hankel transforms.

Firstly we define the h_μ -Lebesgue points for a function $f \in L_{p,\mu}$ as follows. Let $1 \leq p < \infty$ and $f \in L_{p,\mu}$. We will say that $x \in (0, \infty)$ is a h_μ -Lebesgue point for f of order p (to simplify $x \in HL^\mu(f, p)$) if and only if

$$\lim_{t \rightarrow 0^+} \frac{1}{t^{2\mu+2}} \int_0^t |(\tau_y f)(x) - f(x)|^p y^{2\mu+1} dy = 0.$$

It is clear that $HL^\mu(f, p)$ contains to $HL^\mu(f, q)$ provided that $1 \leq p < q < \infty$ and $f \in L_{p,\mu} \cap L_{q,\mu}$.

We now prove that $HL^\mu(f, p)$ is full in $(0, \infty)$.

Proposition 3.1. *Let $1 \leq p < \infty$ and $f \in L_{p,\mu}$. Then the Lebesgue measure of the set $(0, \infty) \setminus HL^\mu(f, p)$ is zero.*

PROOF. Let $t > 0$. Define the set

$$A_t = \left\{ x \in (0, \infty) : \limsup_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y f)(x) - f(x)|^p y^{2\mu+1} dy > t \right\}.$$

The proof will be finish when we prove that $\gamma(A_t) = 0$, where $d\gamma = x^{2\mu+1} dx$.

Let $\epsilon > 0$. We can write $f = g + h$ where g is a smooth function having compact support in $(0, \infty)$ and $\|h\|_{p,\mu} < \epsilon$. By proceeding as in the proof of [3, Proposition 2.1] we can obtain

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y g)(x) - g(x)|^p y^{2\mu+1} dy = 0, \quad x \in (0, \infty).$$

Hence by invoking Jensen's inequality and [13, §2 (2)] we obtain

$$\begin{aligned} (10) \quad & \limsup_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y f)(x) - f(x)|^p y^{2\mu+1} dy \\ & \leq \limsup_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y h)(x) - h(x)|^p y^{2\mu+1} dy \\ & \leq C \left(\limsup_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon (\tau_y |h|^p)(x) y^{2\mu+1} dy + |h(x)|^p \right) \\ & \leq C(M(|h|^p)(x) + |h(x)|^p), \quad x \in (0, \infty). \end{aligned}$$

Here, by M we denote the maximal function introduced by K. STEMPAK [16] defined by

$$M(F)(x) = \sup_{\epsilon > 0} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon \tau_x(|F|)(y) y^{2\mu+1} dy, \quad x \in (0, \infty),$$

when F is a measurable function on $(0, \infty)$.

Then we deduce from (10) that

$$A_t \subset \left\{ x \in (0, \infty) : M(|h|^p)(x) > \frac{t}{2C} \right\} \cup \left\{ x \in (0, \infty) : |h(x)|^p > \frac{t}{2C} \right\}.$$

Moreover, by invoking [17, (3)] we obtain

$$\gamma \left(\left\{ x \in (0, \infty) : M(|h|^p)(x) > \frac{t}{2C} \right\} \right) \leq \frac{C}{t} \|h\|_{p,\mu}^p \leq \frac{C}{t} \epsilon^p.$$

Also, it is clear that

$$\gamma \left(\left\{ x \in (0, \infty) : |h(x)|^p > \frac{t}{2C} \right\} \right) \leq \frac{C}{t} \|h\|_{p,\mu}^p \leq \frac{C}{t} \epsilon^p.$$

Hence, $\gamma(A_t) \leq \frac{C}{t} \epsilon^p$. By letting $\epsilon \rightarrow 0^+$ we can obtain the desired result. $\square$

Our result about strong summability of Hankel transforms is the following.

Proposition 3.2. *Let $-1/2 < \mu < 0$. Let f be in $L_{1,\mu} \cap L_{p,\mu}$, for some $\mu + 2 < p < \infty$, and let x be in $HL^\mu(f, q)$, for some $q > 0$. Then*

$$(11) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |S_\nu(f, \mu; x) - f(x)|^q d\nu = 0.$$

Hence (11) holds for almost every $x \in (0, \infty)$.

PROOF. Without loss of generality we can assume that $\mu + 2 < p < 2$ and that $q = p'$ (see [8, Remark 4]).

Let $T \in (0, \infty)$. By invoking [4, p. 3] we can write, for every $\nu \in (0, T)$,

$$S_\nu(f, \mu; x) - f(x) = I_1(\nu, T) + I_2(\nu, T)$$

where

$$I_1(\nu, T) = \int_0^{1/T} [(\tau_x f)(y) - f(x)] \varphi_\nu(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy$$

and

$$I_2(\nu, T) = \int_{1/T}^\infty [(\tau_x f)(y) - f(x)] \varphi_\nu(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy.$$

Here we have taken into account that $\int_0^\infty \varphi_\nu(y) y^{2\mu+1} dy = 2^\mu \Gamma(\mu+1)$. Recall that $\varphi_\nu(y) = \nu^{2\mu+2} (y\nu)^{-\mu-1} J_{\mu+1}(y\nu)$, $\nu, y \in (0, \infty)$.

Since the function $z^{-\eta}J_\eta(z)$, $\eta \geq -1/2$, is bounded on $(0, \infty)$, it infers that

$$\begin{aligned} & \left\{ \frac{1}{T} \int_0^T |I_1(\nu, T)|^q d\nu \right\}^{1/q} \\ & \leq C \left\{ \frac{1}{T} \int_0^T \left[\nu^{2\mu+2} \int_0^{1/T} |(\tau_x f)(y) - f(x)| \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy \right]^q d\nu \right\}^{1/q} \\ & \leq C \int_0^{1/T} |(\tau_x f)(y) - f(x)| \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy \left\{ \frac{1}{T} \int_0^T \nu^{2(\mu+1)q} d\nu \right\}^{1/q} \\ & = CT^{2\mu+2} \int_0^{1/T} |(\tau_x f)(y) - f(x)| \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy. \end{aligned}$$

Hence, since $HL^\mu(f, q)$ is contained in $HL^\mu(f, 1)$ it concludes that

$$\left\{ \frac{1}{T} \int_0^T |I_1(\nu, T)|^q d\nu \right\}^{1/q} \longrightarrow 0, \quad \text{as } T \rightarrow \infty.$$

On the other hand, according to [12, Theorem 3] we obtain

$$\begin{aligned} & \left\{ \frac{1}{T} \int_0^T |I_2(\nu, T)|^q d\nu \right\}^{1/q} \leq C \left\{ \frac{1}{T} \int_0^T \left| \int_{1/T}^\infty [(\tau_x f)(y) \right. \right. \\ & \quad \left. \left. - f(x)] \nu^{2\mu+2} (y\nu)^{-\mu-1} J_{\mu+1}(y\nu) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy \right|^q d\nu \right\}^{1/q} \\ & \leq CT^{2[(\mu+1)q-\mu-2]/q} \left\{ \int_{1/T}^\infty \left| \frac{(\tau_x f)(y) - f(x)}{y^2} \right|^p y^{2\mu+3} dy \right\}^{1/p}. \end{aligned}$$

We now define the function g through

$$g(t) = \int_0^t |(\tau_x f)(y) - f(x)|^p y^{2\mu+1} dy, \quad t \in (0, \infty).$$

Then by partial integration it obtains

$$\begin{aligned}
 \int_{1/T}^{\infty} \left| \frac{(\tau_x f)(y) - f(x)}{y^2} \right|^p y^{2\mu+3} dy &= \int_{1/T}^{\infty} \frac{g'(y)}{y^{2(p-1)}} dy \\
 &= \frac{g(y)}{y^{2(p-1)}} \Big|_{1/T}^{\infty} + 2(p-1) \int_{1/T}^{\infty} \frac{g(y)}{y^{2p-1}} dy \\
 &= \lim_{y \rightarrow \infty} \frac{1}{y^{2(p-1)}} \int_0^y |(\tau_x f)(s) - f(x)|^p s^{2\mu+1} ds \\
 &\quad - T^{2(p-1)} \int_0^{1/T} |(\tau_x f)(s) - f(x)|^p s^{2\mu+1} ds \\
 &\quad + 2(p-1) \int_{1/T}^{\infty} \frac{g(y)}{y^{2p-1}} dy.
 \end{aligned}$$

Hence since τ_x is a contractive operator from $L_{p,\mu}$ into itself ([17, p. 16]) and since, under our assumption, $HL^\mu(f, q)$ is contained in $HL^\mu(f, p)$, we can conclude that

$$\left\{ \frac{1}{T} \int_0^T |I_2(\nu, T)|^q d\nu \right\}^{1/q} \longrightarrow 0, \quad \text{as } T \rightarrow \infty.$$

Thus the proof is finished. $\square$

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