Publ. Math. Debrecen
55 / 3-4 (1999), 453–464

On the location of the zeros of polynomials defined by linear recursions

By FERENC MÁTYÁS (Eger)

Abstract. Let the polynomials $G_n(x)$ be defined by the recursive formula $G_n(x) = p(x)G_{n-1}(x) + q(x)G_{n-2}(x)$ for $n \geq 2$, where $p(x)$, $q(x)$, $G_0(x)$ and $G_1(x)$ are given polynomials with complex coefficients. The notation $G_n(x) = G_n(p(x), q(x), G_0(x), G_1(x))$ is also used. In this paper we determine the location of the zeros of polynomials $G_n(x)$ if $p(x)$, $q(x)$, $G_0(x)$ and $G_1(x)$ are special polynomials, and give a bound for the absolute values of the complex zeros of the polynomials $G_n(ax+b, q, c, dx+e)$ if $a, b, q, d, e \in \mathbb{C}$ and $aqcd \neq 0$. The theorems generalize some earlier results.

1. Introduction

Let $p(x), q(x), G_0(x)$ and $G_1(x)$ be polynomials with complex coefficients and for $n \geq 2$ let us define the polynomials $G_n(x)$ by

$$G_n(x) = p(x)G_{n-1}(x) + q(x)G_{n-2}(x). \tag{1}$$

We assume that neither of the polynomials $p(x)$ and $q(x)$ is equal to the zero polynomial and at most one of them is constant, furthermore at most one of the polynomials $G_0(x)$ and $G_1(x)$ is the zero polynomial. For brevity we use the notation $G_n(x) = G_n\left(p(x), q(x), G_0(x), G_1(x)\right)$, as well.

With special polynomials we can get the well-known Fibonacci polynomials $(F_n(x))$ and the Chebyshev polynomials of the second kind $(U_n(x))$, namely

$$F_n(x) = G_n(x, 1, 0, 1)$$

Mathematics Subject Classification: 11B39, 12D10.

Key words and phrases: polynomial sequences, zeros of polynomials, bound for the zeros.
Research supported by the Hungarian OTKA Foundation, No. T 020295.

and

$$U_n(x) = G_n(2x, -1, 0, 1).$$

It is known, by trigonometrical identities and $x = \cos\theta$, that

$$U_n(x) = \frac{\sin n\theta}{\sin\theta} \quad (\theta \in \mathbb{C},\ \theta \neq k\pi,\ k \in \mathbb{Z}),$$

and so the zeros z_k of the polynomial $U_n(x)$ are

$$z_k = \cos\frac{k\pi}{n}, \quad k = 1, 2, \dots, n-1. \tag{2}$$

If we consider the polynomials $G_n(x)$ as polynomial functions of $x \in \mathbb{C}$ and H denotes the set of the roots of the equation $p^2(x) + 4q(x) = 0$, then for $x \in \mathbb{C} \setminus H$

$$G_n(x) = a(x)\alpha^n(x) - b(x)\beta^n(x), \tag{3}$$

where $\alpha(x)$ and $\beta(x)$ are the roots of the characteristic equation $\lambda^2 - p(x)\lambda - q(x) = 0$, that is

$$\alpha(x) = \frac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2} \text{ and } \beta(x) = \frac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}, \tag{4}$$

while

$$a(x) = \frac{G_1(x) - \beta(x)G_0(x)}{\alpha(x) - \beta(x)} \quad \text{and} \quad b(x) = \frac{G_1(x) - \alpha(x)G_0(x)}{\alpha(x) - \beta(x)}.$$

Recently, some papers have been published on the zeros of the polynomials $G_n(x)$. These papers can be separated into two classes. One class deals with the real zeros of the Fibonacci-type polynomials $G_n(x, 1, G_0(x), G_1(x))$. For example, Moore [7] investigated the maximal real zero g_n of the polynomials $G_n(x, 1, -1, x-1)$ and proved that $\lim_{n\to\infty} g_n = 3/2$. In [5], under some restrictions, we observed the accumulation points of the set of real zeros of the polynomials $G_n(x, 1, G_0(x), G_1(x))$, while in [4] an asymptotic formula was given for the maximal real zeros of the polynomials $G_n(x, 1, a, x \pm a)$ $(a \in \mathbb{R} \setminus \{0\})$.

The second class of the above-mentioned papers, among others, investigated the complex zeros of the Morgan–Voyce-type polynomials $G_n(x+p, q, G_0(x), G_1(x))$ $(p, q \in \mathbb{R} \setminus \{0\})$. Adopting our notation, Swamy [9], [10] derived explicit formulae for the zeros of the polynomials $G_n(x+2, -1, 1, x+1)$, $G_n(x+2, -1, 1, x+2)$ and $G_n(x+p, -q, 1, x+p \pm \sqrt{q})$. André-Jeannin [1]–[3] determined the zeros of the polynomials $G_n(x+2, -1, 1, x+3)$, $G_n(x+p, -q, 0, 1)$ and $G_n(x+p, -q, 2, x+p)$. These results are based upon the relation between these polynomials and $U_n(x)$.

Using linear-algebraic methods, Ricci [8] proved for the complex zeros z of the polynomials $G_n(x, 1, 1, x+1)$ $(n \geq 1)$ that $|z| < 2$, and a similar result was obtained by us in [6] for the complex zeros of the polynomials $G_n(x, 1, a, x+b)$ $\quad (a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R})$.

The purpose of this paper is to characterize the zeros of the following polynomials: $G_n(p(x), q(x), 0, 1)$, $G_n(p(x), q, c_0, c_1)$ $(q, c_0, c_1 \in \mathbb{C}$, $c_1 = \pm c_0\sqrt{-q})$, $G_n(p(x), q, c, cp(x) + e)$ $(q, c, e \in \mathbb{C}$, $e = 0$ or $\pm c\sqrt{-q} = e)$ and to find a bound for the zeros of the polynomials $G_n(ax + b, q, c, dx + e)$, where $a, b, q, c, d, e \in \mathbb{C}$, $aqcd \neq 0$. From our results one can get the above-mentioned results of Swamy, André-Jeannin, Ricci and Mátyás.

2. Results

Write

$$d_1(x) = \gcd\big(p(x), q(x)\big),$$
$$d_2(x) = \gcd\big(G_1(x), q(x)\big),$$
$$d_3(x) = \gcd\big(G_0(x), G_1(x)\big)$$

and for $x \in \mathbb{C}$ let $\sqrt{x}$ denote one of the complex square roots of x (for example with $0 \leq \arg(\sqrt{x}\,) < \pi$).

It is obvious by (1) that if $d_i(x) = 0$ with some i $(i = 1, 2, 3)$ and a complex $x = z$, then $G_n(z) = 0$ for every $n \geq 2$. In the sequel we do not deal with these simple cases, therefore we suppose that $d_i(x) = 1$ for $i = 1, 2, 3$.

It can easily be derived from (3) that for $n \geq 0$

$$G_n\left(p(x), q(x), 0, G_1(x)\right) = G_1(x) G_n\left(p(x), q(x), 0, 1\right) \tag{5}$$

and for $n \geq 1$, by $\alpha(x)\beta(x) = -q(x)$,

$$G_n\left(p(x), q(x), G_0(x), 0\right) = G_0(x) q(x) G_{n-1}\left(p(x), q(x), 0, 1\right). \tag{6}$$

Since we have supposed that $d_3(x) = 1$ and $d_2(x) = 1$, thus, in (5), $G_1(x)$ is a constant, while, in (6), $G_0(x)$ and $q(x)$ are constants. Therefore, to determine the zeros of the polynomials $G_n(p(x), q(x), 0, G_1(x))$ and $G_n(p(x), q(x), G_0(x), 0)$ is enough to consider the case $G_n(p(x), q(x), 0, 1)$.

Theorem 1. *Let $n \geq 2$. Then $G_n\left(p(x), q(x), 0, 1\right) = 0$ with a complex $x = z$ if and only if z is a root of the equation*

$$p(x) - 2\sqrt{-q(x)}\cos\frac{k\pi}{n} = 0 \tag{7}$$

for some $k = 1, 2, \dots, n-1$.

Remarks. Because of the signs of the cosines, the roots of (7) do not depend on the choice of the square root of $-q(x)$.

By our theorem, to obtain the zeros of $G_n\left(p(x), q(x), 0, 1\right)$ one has to solve $n-1$ equations of type (7), where the degree of these equations does not depend on n.

Using (7), some known results on the zeros of special polynomials can be derived. For instance, let z_k, z'_k and z''_k denote the zeros of the Fibonacci $\bigl(F_n(x)\bigr)$, Pell $\bigl(P_n(x) = G_n\left(2x, 1, 0, 1\right)\bigr)$ and the Jacobsthal polynomial $\bigl(J_n(x) = G_n\left(1, 2x, 0, 1\right)\bigr)$, then

$$z_k = 2i\cos\frac{k\pi}{n}, \quad z'_k = i\cos\frac{k\pi}{n} \quad (k = 1, 2, \dots, n-1)$$

and

$$z''_k = -\frac{1}{8\cos^2\frac{k\pi}{n}} \quad \left(1 \leq k < \frac{n}{2}\right),$$

respectively.

For the polynomial $G_n\left(x+p, -q, 0, 1\right)$ we get that its zeros z'''_k are

$$z'''_k = -p + 2\sqrt{q}\cos\frac{k\pi}{n} \qquad (k = 1, 2, \dots, n-1),$$

as was shown by André-Jeannin in [2].

In the following theorem we characterize the zeros of the polynomials $G_n\left(p(x), q, c_0, c_1\right)$, where c_0 and c_1 are special constants.

Theorem 2. *Let $q, c_0, c_1 \in \mathbb{C} \setminus \{0\}$, $c_1 = \pm c_0\sqrt{-q}$ and $n \geq 2$. For a complex number $x = z$, in the case $c_1 = c_0\sqrt{-q}$, $G_n\left(p(x), q, c_0, c_1\right) = 0$ if and only if z satisfies the equation*

$$p(x) - 2\sqrt{-q}\cos\frac{2k-1}{2n-1}\pi = 0,$$

while, in the case $c_1 = -c_0\sqrt{-q}$, z satisfies the equation

$$p(x) - 2\sqrt{-q}\cos\frac{2k}{2n-1}\pi = 0$$

for some $k = 1, 2, \ldots, n-1$.

Considering the zeros of the polynomials $G_n\left(p(x), q, c, cp(x) + e\right)$, where e and c are special contants, we have:

Theorem 3. *Let $n \geq 1$, $q, c \in \mathbb{C} \setminus \{0\}$, $e \in \mathbb{C}$, $e = 0$ or $\pm c\sqrt{-q} = e$. The zeros of the polynomial $G_n\left(p(x), q, c, cp(x) + e\right)$ are equivalent to the roots of the following equations for some $k = 1, 2, \ldots, n$:*
in the case $e = 0$

$$p(x) - 2\sqrt{-q}\cos\frac{k\pi}{n+1} = 0,$$

in the case $-c\sqrt{-q} = e$

$$p(x) - 2\sqrt{-q}\cos\frac{2k-1}{2n+1}\pi = 0$$

and in the case $c\sqrt{-q} = e$

$$p(x) - 2\sqrt{-q}\cos\frac{2k}{2n+1}\pi = 0.$$

Remark. The mentioned results on the zeros of the Morgan–Voyce-type polynomials follow from Theorem 3 if we substitute the actual polynomials. For example the zeros $x = z_k$ of $G_n(x + p, -q, 1, x + p + \sqrt{q})$ are

$$z_k = -p + 2\sqrt{q}\cos\frac{2k}{2n+1}\pi \quad (k = 1, 2, \ldots, n),$$

since in this case $c\sqrt{-q} = e$.

Moreover, using linear-algebraic methods, we derive a bound for the zeros of a general class of polynomials $G_n(ax+b, q, c, dx+e)$. The following theorem generalizes the result of [6].

Theorem 4. *Let $a, b, q, c, d, e \in \mathbb{C}$, $aqcd \neq 0$ and $n \geq 1$. If $G_n(ax + b, q, c, dx + e) = 0$ for $x = x_1, x_2, \ldots, x_n$, then*

$$\max_{1\leq i\leq n} |x_i| \leq \frac{1}{|ad|}\left(\max\bigl(|ca\sqrt{q}| + |ae - db|,\ 2|d\sqrt{q}|\bigr) + |bd|\right).$$

Remark. According to Theorem 4, for example the zeros of the Fermat–Lucas polynomials $G_n(3x,-2,2,3x)$ satisfy the inequality $|z|\le 2\sqrt{2}/3$ for every $n\ge 1$.

3. Lemmas and proofs

To prove our theorems some auxiliary results are needed.

Lemma 1. *Let $G_n(x)=G_n\left(p(x),q(x),G_0(x),G_1(x)\right)$ and the degree of $q(x)\ge 1$. If $q(z)=0$ with a complex z, then $G_n(z)\ne 0$ for every $n\ge 1$.*

Proof. By the assumption $d_2(x)=1$ we have $G_1(z)\ne 0$. If there is an $n\ge 2$ for which $G_n(z)=0$, then (1) and $d_1(x)=1$ imply $G_{n-1}(z)=0$, but this leads to $G_1(z)=0$, which is a contradiction.

According to Lemma 1, the zeros of the polynomial $q(x)$ can be omitted at the investigation of zeros of the polynomial $G_n(x)$. Let $K=\{z:z\in\mathbb{C}, q(z)=0\}$ and H is as before, that is, $H=\{z:z\in\mathbb{C},\ p^2(z)+4q(z)=0\}$.

Lemma 2. *For $x\in\mathbb{C}\setminus(H\cup K)$ and $n\ge 1$ we have*

$$\begin{aligned}&G_n(p(x),q(x),G_0(x),G_1(x))\\&\qquad=\left(\sqrt{\pm q(x)}\right)^{n-1}G_n\left(\frac{p(x)}{\sqrt{\pm q(x)}},\pm 1,\sqrt{\pm q(x)}G_0(x),G_1(x)\right),\end{aligned}$$

where the same signs are taken together.

Proof. By (4),

$$\alpha(x)=\frac{p(x)+\sqrt{p^2(x)+4q(x)}}{2}=\sqrt{\pm q(x)}\,\frac{\frac{p(x)}{\sqrt{\pm q(x)}}+\sqrt{\left(\frac{p(x)}{\sqrt{\pm q(x)}}\right)^2\pm 4}}{2}$$

and

$$\beta(x)=\frac{p(x)-\sqrt{p^2(x)+4q(x)}}{2}=\sqrt{\pm q(x)}\,\frac{\frac{p(x)}{\sqrt{\pm q(x)}}-\sqrt{\left(\frac{p(x)}{\sqrt{\pm q(x)}}\right)^2\pm 4}}{2}.$$

The equation $\lambda^2 - \frac{p(x)}{\sqrt{\pm q(x)}}\lambda - (\pm 1) = 0$ is the characteristic equation of the polynomials $G_n\left(\frac{p(x)}{\sqrt{\pm q(x)}}, \pm 1, \sqrt{\pm q(x)}G_0(x), G_1(x)\right)$ and let $\alpha^\star(x)$ and $\beta^\star(x)$ denote the roots of it. Then

$$\alpha^\star(x)\sqrt{\pm q(x)} = \alpha(x), \quad \beta^\star(x)\sqrt{\pm q(x)} = \beta(x)$$

and (3) yield

$$\begin{aligned}
&G_n(p(x), q(x), G_0(x), G_1(x)) \\
&\quad= \frac{G_1(x) - \sqrt{\pm q(x)}G_0(x)\beta^\star(x)}{\sqrt{\pm q(x)}(\alpha^\star(x) - \beta^\star(x))}\left(\sqrt{\pm q(x)}\right)^n \alpha^{\star n}(x) \\
&\qquad- \frac{G_1(x) - \sqrt{\pm q(x)}G_0(x)\alpha^\star(x)}{\sqrt{\pm q(x)}(\alpha^\star(x) - \beta^\star(x))}\left(\sqrt{\pm q(x)}\right)^n \beta^{\star n}(x) \\
&\quad= \left(\sqrt{\pm q(x)}\right)^{n-1} G_n\left(\frac{p(x)}{\sqrt{\pm q(x)}}, \pm 1, \sqrt{\pm q(x)}G_0(x), G_1(x)\right).
\end{aligned}$$

The next lemma shows a relation between the polynomials $U_n(x)$ and $G_n(2x, -1, 1, t)$, where $t \in \mathbb{C} \setminus \{0\}$.

Lemma 3. *For $n \geq 1$ and $t \in \mathbb{C} \setminus \{0\}$*

$$G_n(2x, -1, 1, t) = tU_n(x) - U_{n-1}(x).$$

Proof. It is easy to verify that $G_1(2x, -1, 1, t) = t = tU_1(x) - U_0(x)$ and $G_2(2x, -1, 1, t) = 2xt - 1 = tU_2(x) - U_1(x)$. Furthermore, we suppose that the statement is true for $n-1$ and $n-2$ ($n \geq 3$) then, by (1) and our induction hipothesis,

$$\begin{aligned}
G_n(2x, -1, 1, t) &= 2xG_{n-1}(2x, -1, 1, t) - G_{n-2}(2x, -1, 1, t) \\
&= 2x\big(tU_{n-1}(x) - U_{n-2}(x)\big) - \big(tU_{n-2}(x) - U_{n-3}(x)\big) \\
&= t\big(2xU_{n-1}(x) - U_{n-2}(x)\big) - \big(2xU_{n-2}(x) - U_{n-3}(x)\big) \\
&= tU_n(x) - U_{n-1}(x).
\end{aligned}$$

To prove Theorem 4 we need the following lemma.

Lemma 4. *Let $n \geq 1$ and $a, b \in \mathbb{C}$ $(a \neq 0)$. If $G_n(x, 1, a, x+b) = 0$ for the complex numbers $x = x_1, x_2, \dots, x_n$, then*

$$\max_{1 \leq i \leq n} |x_i| \leq \max\big(|a| + |b|, 2\big)$$

for every $n \geq 1$.

Proof. The proof of this lemma can be found in [6]. An outline of the proof is as follows. First one can verify by induction on n that the polynomial $G_n(x, 1, a, x+b)$ is the characteristic polynomial of the $n \times n$ matrix

$$\mathbf{A}_n = \begin{pmatrix} -b & -ai & 0 & \cdots & 0 & 0 & 0 \\ -i & 0 & -i & \cdots & 0 & 0 & 0 \\ 0 & -i & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -i & 0 & -i \\ 0 & 0 & 0 & \cdots & 0 & -i & 0 \end{pmatrix}.$$

Therefore, the roots of $G_n(x, 1, a, x+b) = 0$ are the eigenvalues of the matrix $\mathbf{A}_n$. We get, by Gershgorin's theorem, that the eigenvalues (roots) $x_1, x_2, \dots, x_n$ are in or on the so-called Gershgorin circles. In our case there are only two distinct circles (with distinct midpoints). Their midpoints are $-b$ and 0 in the Gaussian plane, while their radii are $|a|$ and 2, respectively. From this the inequality

$$\max_{1 \leq i \leq n} |x_i| \leq \max\big(|a| + |b|, 2\big)$$

follows immediately for every $n \geq 1$.

Proof of Theorem 1. Using Lemma 2, we get that for $x \in \mathbb{C} \setminus (H \cup K)$

$$G_n(p(x), q(x), 0, 1) = \left(\sqrt{-q(x)}\right)^{n-1} G_n\left(\frac{p(x)}{\sqrt{-q(x)}}, -1, 0, 1\right).$$

Let

$$2y = \frac{p(x)}{\sqrt{-q(x)}}, \tag{8}$$

then

$$G_n\left(\frac{p(x)}{\sqrt{-q(x)}}, -1, 0, 1\right) = G_n(2y, -1, 0, 1) = U_n(y).$$

By (2), $U_n(y_k) = 0$ if and only if $y_k = \cos\frac{k\pi}{n}$ $(k = 1, 2, \dots, n-1)$. Therefore, with (8), the zeros of the polynomial $G_n(p(x), q(x), 0, 1)$ $(n \ge 2)$ satisfy the equation $p(x) - 2\sqrt{-q(x)}\cos\frac{k\pi}{n} = 0$ for some $k = 1, 2, \dots, n-1$.

Proof of Theorem 2. Acccording to Lemma 2 and (3), for $n \ge 2$ we obtain

$$G_n(p(x), q, c_0, c_1) = \left(\sqrt{-q}\right)^{n-1} G_n\left(\frac{p(x)}{\sqrt{-q}}, -1, c_0\sqrt{-q}, c_1\right)$$
$$= \left(\sqrt{-q}\right)^{n} c_0 G_n\left(\frac{p(x)}{\sqrt{-q}}, -1, 1, \frac{c_1}{c_0\sqrt{-q}}\right).$$

With

$$2y = \frac{p(x)}{\sqrt{-q}}, \tag{9}$$

one can see that

$$G_n(p(x), q, c_0, c_1) = \left(\sqrt{-q}\right)^{n} c_0 G_n\left(2y, -1, 1, \frac{c_1}{c_0\sqrt{-q}}\right),$$

from which, by Lemma 3,

$$G_n(p(x), q, c_0, c_1) = \left(\sqrt{-q}\right)^{n} c_0 \left(\frac{c_1}{c_0\sqrt{-q}} U_n(y) - U_{n-1}(y)\right)$$

follows. Therefore, $G_n(p(x), q, c_0, c_1) = 0$ if and only if

$$\frac{c_1}{c_0\sqrt{-q}} U_n(y) = U_{n-1}(y),$$

hence, with $y = \cos\theta$ $\big(\theta \in \mathbb{C} \setminus \{k\pi \,:\, k \in \mathbb{Z}\}\big)$, we get

$$\frac{c_1}{c_0\sqrt{-q}} \frac{\sin n\theta}{\sin\theta} = \frac{\sin(n-1)\theta}{\sin\theta}. \tag{10}$$

In our cases, (10) can easily be solved for every $n \ge 2$. That is, if $c_1 = c_0\sqrt{-q}$ then $\theta = \frac{2k-1}{2n-1}\pi$ $\;(k \in \mathbb{Z})$ are the solutions of (10), and so we get the distinct values y_k by

$$y_k = \cos\frac{2k-1}{2n-1}\pi \quad (k = 1, 2, \dots, n-1).$$

If $c_1 = -c_0\sqrt{-q}$ then the solutions of (10) are $\theta = \frac{2k}{2n-1}\pi \quad (k \in \mathbb{Z})$, and so the distinct values y_k are

$$y_k = \cos\frac{2k}{2n-1}\pi \quad (k = 1, 2, \dots, n-1).$$

Using (9), the desired formulae can be obtained.

Proof of Theorem 3. It is easy to see by (1) that for $n \geq 1$

$$G_n(p(x), q, c, cp(x) + e) = G_{n+1}\left(p(x), q, \frac{e}{q}, c\right).$$

If $e = 0$ then, by Theorem 1, the zeros of $G_n(p(x), q, c, cp(x))$ and

$$p(x) - 2\sqrt{-q}\cos\frac{k\pi}{n+1} = 0$$

coincide for some $k = 1, 2, \dots, n$.

If $-c\sqrt{-q} = e$ or $c\sqrt{-q} = e$ then, by Theorem 2, the zeros of the polynomial $G_n(p(x), q, c, cp(x) + e)$ and the roots of the equations

$$p(x) - 2\sqrt{-q}\cos\frac{2k-1}{2n+1}\pi = 0$$

or

$$p(x) - 2\sqrt{-q}\cos\frac{2k}{2n+1}\pi = 0$$

are the same for some $k = 1, 2, \dots, n$, respectively.

Proof of Theorem 4. By Lemma 2,

$$G_n(ax + b, q, c, dx + e) = (\sqrt{q}\,)^{n-1}\, G_n\left(\frac{ax+b}{\sqrt{q}}, 1, c\sqrt{q}, dx + e\right).$$

With

$$y = \frac{ax+b}{\sqrt{q}} \qquad \left(x = \frac{y\sqrt{q} - b}{a}\right), \tag{11}$$

we get

$$G_n(ax + b, q, c, dx + e) = (\sqrt{q}\,)^{n-1}\, G_n\left(y, 1, c\sqrt{q}, \frac{d\sqrt{q}}{a}y + \frac{ae - db}{a}\right),$$

from which, by (3),

$$G_n(ax+b,q,c,dx+e) = (\sqrt{q}\,)^n \ \frac{d}{a} G_n\left(y,1,\frac{ca}{d},y+\frac{ae-db}{d\sqrt{q}}\right)$$

follows. According to Lemma 4, the roots $y_1, y_2, \dots, y_n$ of the equation

$$G_n\left(y,1,\frac{ca}{d},y+\frac{ae-db}{d\sqrt{q}}\right) = 0$$

satisfy the inequality

$$\max_{1\le i\le n} |y_i| \le \max\left(\left|\frac{ca}{d}\right| + \left|\frac{ae-db}{d\sqrt{q}}\right|,\ 2\right)$$

for every $n \ge 1$. That is, by (11),

$$\begin{aligned}\max_{1\le i\le n} |x_i| &\le \left|\frac{\sqrt{q}}{a}\right| \max_{1\le i\le n} |y_i| + \left|\frac{b}{a}\right| \\ &\le \left|\frac{\sqrt{q}}{a}\right| \max\left(\left|\frac{ca}{d}\right| + \left|\frac{ae-db}{d\sqrt{q}}\right|,\ 2\right) + \left|\frac{b}{a}\right| \\ &= \frac{1}{|ad|}\Big(\max\big(|ac\sqrt{q}| + |ae-bd|,\ 2|d\sqrt{q}|\big) + |bd|\Big).\end{aligned}$$

This completes the proof.

Acknowledgements. The author would like to thank Professors Péter Kiss and Béla Brindza for their helpful comments on an earlier draft of the manuscript. The author also expresses his gratitude to the anonymous referee for the valuable remarks.

References

[1] R. André-Jeannin, A generalization of Morgan-Voyce polynomials, *The Fibonacci Quarterly* **32.2** (1994), 228–231.

[2] R. André-Jeannin, A note on the general class of polynomials, *The Fibonacci Quarterly* **32.5** (1994), 445–454.

[3] R. André-Jeannin, A note on the general class of polynomials, Part II, *The Fibonacci Quarterly* **33.4** (1995), 341–351.

[4] F. Mátyás, The asymptotic behavior of real roots of Fibonacci-like polynomials, *Acta Academiae Paedagogicae Agriensis (Sectio Matematicae)* **24** (1997), 55–61.

[5] F. Mátyás, Real roots of Fibonacci-like polynomials (Győry, Pethő and Sós, eds.), Number Theory, *Walter de Gruyter GmbH & Co.*, *Berlin, New York*, 1998, 361–370.

[6] F. Mátyás, Bounds for the zeros of Fibonacci-type polynomials, *Acta Academiae Paedagogicae Agriensis (Sectio Matematicae)* **25** (1998), 17–23.

[7] G. A. Moore, The limit of the golden numbers is 3/2, *The Fibonacci Quarterly* **32.3** (1994), 211–217.

[8] P. E. Ricci, Generalized Lucas polynomials and Fibonacci polynomials, *Rivista di Matematica Univ. Parma* **4** no. 5 (1995), 137–146.

[9] M. N. S. Swamy, Further properties of the polynomials defined by Morgan-Voyce, *The Fibonacci Quarterly* **6.5** (1968), 166–175.

[10] M. N. S. Swamy, On a class of generalized polynomials, *The Fibonacci Quarterly* **35.4** (1997), 329–334.

FERENC MÁTYÁS
DEPARTMENT OF MATHEMATICS
ESZTERHÁZY KÁROLY TEACHERS' TRAINING COLLEGE
H–3301 EGER, P.O.B. 43
HUNGARY

E-mail: matyas@gemini.ektf.hu

(Received May 5, 1998; revised December 31, 1998)