

Perfect measures and equicontinuity

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Abstract. In this paper we give a characterization of the equicontinuous subsets of the space $M_p(X, E')$ of vector-valued perfect measures on a completely regular topological space X generalizing the result of KOUMOULLIS [2] for the scalar case.

Let X be a completely regular Hausdorff space, E a normed space over K , the field of real or complex numbers, and $C_b(X, E)$ the space of all continuous E -valued functions on X . As in [2] $M_p(X)$ denotes the space of all scalar-valued Baire perfect measures on X , which is the dual of the space $C_b(X)$ with the topology β_p . For a real valued continuous function f defined on X , $f^\sim$ is its continuous extension to βX . For $\mu \in (C_b(X), \|\cdot\|')$, $\mu^\sim : C(\beta X) \rightarrow K$ is defined by $\mu^\sim(f^\sim) = \mu(f|_X)$. A subset G in a completely regular Hausdorff space Y will be called *distinguished* if there exist a continuous mapping ϕ from Y onto a separable metric space such that $G = \phi^{-1}(\phi(G))$. The class of all distinguished subsets of $\beta(X)$ disjoint from X will be denoted by $D(\beta X)$. For D in $D(\beta X)$, the topology γ_D , on $C_b(X, E)$ is defined to be the one generated by the seminorms $\|\cdot\|_g$, as g varies over the set $B_D(X)$ of all bounded scalar-valued functions on βX , vanishing at infinity and zero on D , $\|f\|_g = \sup_{x \in \beta(X)} (\|f(x)\| \wedge |g(x)|)$. As in [3, Theorem 2.4] it can be verified that γ_D is the finest locally convex topology agreeing with the topology of uniform convergence on compact subsets of $\beta(X \setminus D)$, on norm-bounded subsets of $C_b(X, E)$. The topology β_p on $C_b(X, E)$ is defined in [1] as $\beta_p = \bigwedge \{\gamma_D : D \in D\}$. A locally convex topology on $C_b(X, E)$ will be called *locally solid* if it has

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a 0-nbd base consisting of absolutely convex sets V , such that $f \in V$ and $\|g\| \leq \|f\|$ implies $g \in V$. It is proved in [1, Theorem 5] that the topology β_p on $C_b(X, E)$ is locally solid. For a duality $\langle F, G \rangle$, $A \subset F$ the polar of A is defined as $A^o = \{g \in G, |\langle a, g \rangle| \leq 1, \text{ for every } a \in A\}$. We define $M_p(X, E')$ as $M_p(X, E') = \{\mu \in M_\sigma(X, E), \mu_x \in M_p(X), \text{ for every } x \in E\}$, here μ_x is defined as $\mu_x(f) = \mu(f \otimes x)$. In [1, Theorem 8] it was proved that if $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$ then the dual of $(C_b(X, E), \beta_p)$ is $M_p(X, E')$. In $C_b(X)$, the space of all bounded real-valued continuous functions defined on X , has also been defined the strict topology β_o which yield as a dual space a subspace of $M_p(X)$ which is the space $M_t(X)$ of tight Baire measures [3] and it is well known that $M_p(X) = M_t(X)$ when X is a separable metric space [4]. Also a subset H of $M_t(X)$ is β_o equicontinuous if H is norm bounded and for every $\epsilon > 0$ there is a compact subset K of X such that $|\mu|_*(X \setminus (K)) < \epsilon$ for every $\mu \in H$ [4].

The main purpose of this work is to extend to the vector case the characterization of equicontinuity of subsets of scalar perfect measures given by KOUMOULLIS in [2].

Lemma. *Let X and Y be completely regular spaces, $\varphi : X \rightarrow Y$ a continuous map, E a normed vector space, $L \in E'$ and $e \in E$. We define the maps*

$$(C_b(Y), \beta_p) \xrightarrow{\psi_e} (C_b(Y, E), \beta_p) \xrightarrow{T_\varphi} (C_b(X, E), \beta_p) \xrightarrow{\Phi_L} (C_b(X), \beta_p)$$

by $\Psi_e(g) = g \otimes e$, $T_\varphi(h) = h \circ \varphi$ and $\Phi_L(h) = L \circ h$. Then we have the following:

1. Ψ_e , T_φ and Φ_L are continuous.
2. If $L(e) = 1$, $\Phi_L \circ T_\varphi \circ \Psi_e = T_\varphi^*$ where T_φ^* is given by $T_\varphi^*(g) = g \circ \varphi$
3. If $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$ and H is a β_p -equicontinuous subset of $M_p(X, E')$, then $\{|\mu| \circ \Phi_L : \mu \in H\}$ is an β_p -equicontinuous subset of $M_p(X, E')$ and $\{|\mu| \circ T_\varphi^* : \mu \in H\}$ is an β_p -equicontinuous subset of $M_p(Y)$

PROOF.

1. The continuity of T_φ is Theorem 3 in [1]. The proofs of the continuity of Ψ_e and Φ_L are just easy applications of Corollary 6 in [1].

2. For any $g \in C_b(Y)$ we have that:

$$\begin{aligned}\Phi_L \circ T_\varphi \circ \Psi(g) &= \Phi_L(T_\varphi(g \otimes e)) = \Phi_L((g \otimes e) \circ \varphi) \\ &= (L \circ (g \otimes e) \circ \varphi) = g \circ \varphi = T_\varphi^*(g)\end{aligned}$$

therefore the equality follows.

3. From Theorem 1 in [1] we have that $|\mu| \in M_p(X)$ for every $\mu \in M_p(X, E')$. Thus the set $\{|\mu| \circ \Phi_L : \mu \in H\}$ is included in $M_p(X, E')$ since Φ_L is continuous. To prove the equicontinuity of this set we can assume without loss of generality that $\|L\| \leq 1$. Since H is β_p -equicontinuous, there exists an absolutely convex solid O -neighborhood W in $(C_b(X, E), \beta_p)$ such that $W \subset H^\circ$. Now for every $f \in W$ and $\mu \in H$,

$$\begin{aligned}|\mu| \circ \Phi_L(f) &= |\mu|(L \circ f) = \sup \{|\mu(h)| : h \in C_b(X, E), \|h\| \leq |L \circ f|\} \\ &\leq \sup \{|\mu(h)| : h \in C_b(X, E), \|h\| \leq \|f\|\} \\ &\leq \sup \{|\mu(h)| : h \in W\} \leq 1,\end{aligned}$$

where the first inequality follows from $|L \circ f| \leq \|L\| \leq \|f\|$ and the second from the fact that W is solid. Therefore $\{|\mu| \circ \Phi_L : \mu \in H\}$ is β_p -equicontinuous. For the proof of the equicontinuity of the set $\{|\mu| \circ T_\varphi^* : \mu \in H\}$ we choose L and e so that $L(e) = 1$. As before we observe that this set is included in $M_p(Y)$. Since

$$T_\varphi \circ \Psi_e : (C_b(Y), \beta_p) \longrightarrow (C_b(X, E), \beta_p)$$

is a continuous operator (by (1)) and

$$\{|\mu| \circ T_\varphi^* : \mu \in H\} = \{|\mu| \circ \Phi_L \circ T_\varphi \circ \Psi_e : \mu \in H\}$$

(by (2)), the conclusion follows from the equicontinuity of $\{|\mu| \circ \Phi_L : \mu \in H\}$ □

Main Theorem

Theorem. Suppose $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$. A subset H of $M_p(X, E')$ is β_p -equicontinuous if and only if: (a) H is norm bounded and (b) for any continuous function f from X onto a separable metric

space Y and every $\epsilon > 0$ there is a compact subset K of Y such that $|\mu|(X \setminus f^{-1}(K)) < \epsilon$ for every $\mu \in H$.

PROOF. Let $\phi : X \mapsto Y$ be a continuous map from X onto the separable metric space Y and H an equicontinuous subset of $M_p(X, E')$. Then by the Lemma, $\{|\mu| \circ T_\phi^* : \mu \in H\}$ is an β_o -equicontinuous subset of $M_t(Y)$, and by Theorem 7.15 [3] for any $\epsilon \geq 0$ there exists a compact subset $K \subseteq Y$ such that $|\mu|(X \setminus \phi^{-1}(K)) \leq \epsilon$ for every $\mu \in H$. Also, since $\beta_p \leq \|\cdot\|$, H is norm bounded.

On the other hand, if $H \subseteq M_p(X, E')$ satisfies (a) and (b). Put $\alpha_o = \sup \{|\mu| : \mu \in H\}$. To prove that H is β_p -equicontinuous is equivalent to prove that H is γ_D -equicontinuous for every $D \in D(\beta X)$. Also since γ_D is the finest locally convex topology agreeing with the topology of uniform convergence on compact subsets of $\beta X \setminus D$, on norm-bounded subsets of $C_b(X, E)$ it is enough to prove that for any $k > 0$, there exist a compact subset $K_D \subseteq \beta X \setminus D$ and some $\eta > 0$ such that the set

$$K = \left\{ f \in C_b(X, E) : \|f\| \leq k, \|f\|_{\widetilde{\|\cdot\|_{K_D}}} < \eta \right\}$$

is contained in the set

$$\{g \in C_b(X, E) : |\mu(g)| \leq 1, \mu \in H\}.$$

Take $\eta = 1/(2 + 2\alpha_o)$. Since D is a distinguished subset of βX disjoint from X there exist a continuous perfect map ρ from βX onto a compact metric space M such that $D = \rho^{-1}(\rho(D))$. Since $\rho(X)$ is a separable metric space it follows from (b) that there exists a compact subset K of $\rho(X)$ such that $|\mu|(X \setminus \rho^{-1}(K)) < 1/(2k + 1)$ for every $\mu \in H$. Take $K_D = \rho^{-1}(K)$. Then K_D is a compact subset of βX and is included in $\beta X \setminus D$ because $D = \rho^{-1}(\rho(D))$ and D is disjoint from X . Therefore we have that

$$|\mu|^\sim[(\beta X \setminus D) \setminus K_D] < 1/(2k + 1).$$

Then if $f \in K$ and $\mu \in H$ we have

$$\begin{aligned} |\mu|(f) &\leq \int \|f\| d|\mu| = \int \|f\|^\sim d|\mu|^\sim \\ &\leq \int_{K_D} \|f\|^\sim d|\mu|^\sim + \int_{(\beta X \setminus D) \setminus K_D} \|f\|^\sim d|\mu|^\sim \\ &\leq \alpha_o/(2 + 2\alpha_o) + k/(2k + 1) \leq 1 \end{aligned}$$

this proves that H is β_p -equicontinuous. $\square$

Corollary. Suppose $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$. A subset H of $M_p(X, E')$ is β_p -equicontinuous if and only if the set $\{|\mu| : \mu \in H\}$ is a β_p -equicontinuous subset of $M_p(X)$

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References

- [1] S. S. KHURANA and JORGE VIELMA, Strict topology and perfect Measures, *Czech. Math. J.* **40** (115) (1990), 1–7.
- [2] G. KOUMOULLIS, Perfect, u -additive measures, and strict topologies, *Illinois J. Math.* **26** (1982), 466–478.
- [3] F. D. SENTILLES, Bounded continuous functions on completely regular spaces, *Trans. Amer. Math. Soc.* **168** (1972), 311–336.
- [4] ROBERT F. WHEELER, A survey of Baire measures and strict topologies, *Expo. Math.* **2** (1983), 97–190.

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