Publ. Math. Debrecen
56 / 1-2 (2000), 21–32

Horizontal lifts in the higher order geometry

By IOAN BUCATARU (Iaşi)

Abstract. First, using the complete lift of a linear connection we construct the horizontal lift of a vector field with the aid of an arbitrary semispray S. It is proved that this horizontal lift is independent on the choice of the semispray S. This reformulates well-known constructions for the case of tangent bundle, [4], [8].

Secondly, the complete and horizontal lifts of vector fields are constructed for the tangent bundle of second order. In this new framework we have a horizontal lift and two vertical lifts. The nonlinear connection associated to the horizontal lift is proven to be just Miron's nonlinear connection, [6]. Thirdly, the above notions and constructions are given for the tangent bundle of order $k > 2$.

1. The horizontal lift to the tangent bundle

Let (TM, π, M) be the tangent bundle of a real, smooth, n-dimensional manifold M. For a local chart $(U, \phi = (x^i))$ on M, its induced local chart on TM will be denoted by $(\pi^{-1}(U), \Phi = (x^i, y^i))$. In a point $u = (x, y) \in TM$, the natural basis of the tangent space T_uTM is denoted by $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^i}|_u\}$. The linear map induced by the canonical submersion $\pi : TM \to M$ is denoted by $\pi_{*,u} : T_uTM \to T_{\pi(u)}M$, $u \in TM$. For each $u \in TM$, $V_uTM = \operatorname{Ker} \pi_{*,u}$ is an n-dimensional vector subspace of the space T_uTM, and a basis of it is $\{\frac{\partial}{\partial y^i}|_u\}$. The map $VTM : u \in TM \mapsto V_uTM \subset T_uTM$ is a regular, n-dimensional and integrable distribution, called the vertical distribution.

The tensor field $J = \frac{\partial}{\partial y^i} \otimes dx^i$, is globally defined on TM and is called the natural almost tangent structure.
One has: 1. $J^2 = 0$, 2. $\operatorname{Im} J = \operatorname{Ker} J = VTM$, 3. $\operatorname{rank} J = n$.

Mathematics Subject Classification: 53C60, 58A20, 58A30.
Key words and phrases: semispray, horizontal lift, nonlinear connection.
Partially supported by the Grants of the Romanian Academy.

The vector field $\overset{1}{\Gamma} = y^i \frac{\partial}{\partial y^i}$, globally defined on TM, is called the *Liouville vector field.* A vector field $S \in \chi(TM)$ is called a *semispray* on TM if and only if $JS = \overset{1}{\Gamma}$. It follows that

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{1.1}$$

where the functions G^i are defined on the domain of local charts.

For a vector field $X = X^i \frac{\partial}{\partial x^i} \in \chi(M)$ we denote its vertical lift by $X^v = (X^i \circ \pi) \frac{\partial}{\partial y^i}$. The map $l_v : \chi(M) \to \chi(TM)$, defined by $l_v(X) = X^v$ is $\mathcal{F}(M)$-linear and is called also the *vertical lift.*

For $X = X^i(x) \frac{\partial}{\partial x^i} \in \chi(M)$, the vector field $X^c \in \chi(TM)$ defined by

$$X^c(x,y) = X^i(x) \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j}(x) y^j \frac{\partial}{\partial y^i}$$

is called the *complete lift of* X. We observe that if $X = X^i \frac{\partial}{\partial x^i} \in \chi(M)$ and S is a semispray on TM, then

$$X^c = (X^i \circ \pi) \frac{\partial}{\partial x^i} + S(X^i \circ \pi) \frac{\partial}{\partial y^i}.$$

So the complete lift can be constructed using a semispray and the result does not depend on the choice of the semispray.

For the vertical and the complete lifts the next formulae hold:

$$J(X^c) = X^v, \quad (fX)^c = S(f) X^v + f X^c, \quad X \in \chi(M),\ f \in \mathcal{F}(M).$$

The map $X \in \chi(M) \mapsto X^c \in \chi(TM)$ is not an $\mathcal{F}(M)$-linear map. Next we shall introduce a modification of this map such that the new map will be $\mathcal{F}(M)$-linear.

Definition 1.1. An $\mathcal{F}(M)$-linear map $l_h : \chi(M) \to \chi(TM)$ for which we have

$$J \circ l_h = l_v, \tag{1.2}$$

is called a horizontal lift.

A subbundle HTM of the tangent bundle (TTM, π_{TM}, TM) which is supplementary to the vertical subbundle, i.e. the following Whitney sum holds

$$TTM = HTM \oplus VTM, \tag{1.3}$$

is called a nonlinear connection on TM. A nonlinear connection determines an n-dimensional distribution $N : u \in TM \to N_u = H_u TM \subset T_u TM$.

We have that every horizontal lift l_h determines a nonlinear connection $N = \operatorname{Im} l_h$ on TM. Conversely, every nonlinear connection N on TM determines a horizontal lift l_h which is the inverse of the map $\pi_{*,u}|_{N_u} : N_u \to T_{\pi(u)}M$.

Let D be a linear connection, with the local coefficients γ^i_{jk}. One can associate to D ([8], Ch. I, §6) a unique linear connection D^c on TM, called the complete lift of D, which satisfies:

$$D^c_{X^c} Y^c = (D_X Y)^c.$$

For D^c the following formulae hold:

$$D^c_{X^c} Y^v = D^c_{X^v} Y^c = (D_X Y)^v, \quad D^c_{X^v} Y^v = 0. \tag{$*$}$$

The linear connection D^c preserves the vertical distribution by parallelism, i.e. for a vertical vector field Y and $X \in \chi(TM)$ we have that $D^c_X Y$ is a vertical vector field.

Proposition 1.1. *Let S be a semispray on TM. For a vector field $X \in \chi(M)$ we define $X^h \in \chi(TM)$ by*

$$X^h = X^c - D^c_S X^v. \tag{1.4}$$

The map $l_h : \chi(M) \to \chi(TM)$ defined by $l_h(X) = X^h$ is a horizontal lift and it does not depend on the choice of the semispray S.

Proof. First we prove that the map (l_h) is $\mathcal{F}(M)$-linear. For $f \in \mathcal{F}(M)$ we have $(fX)^c = (f \circ \pi)X^c + S(f)X^v$ and $(fX)^v = (f \circ \pi)X^v$. It follows $(fX)^h = (f \circ \pi)X^c + S(f)X^v - S(f)X^v - (f \circ \pi)D^c_S X^v = (f \circ \pi)X^h$.

Now we prove that $J \circ l_h = l_v$. We have $J(X^c) = X^v$, $\forall X \in \chi(M)$. Since $\operatorname{Ker} J = V$ and $D^c_S X^v$ is a vertical vector field, it results that $J(D^c_S X^v) = 0$. Consequently $(J \circ l_h)(X) = J(X^c) - J(D^c_S X^v) = J(X^c) = X^v = l_v(X)$, $\forall X \in \chi(M)$.

It remains to prove that l_h does not depend on the choice of the semispray S. Let S_1 and S_2 be two semisprays on TM and X^{h_1}, X^{h_2} the horizontal lifts of the vector field $X \in \chi(M)$ constructed with S_1 and S_2, respectively. We have $X^{h_1} - X^{h_2} = D^c_{S_2 - S_1} X^v$. Since $S_1 - S_2$ and X^v are vertical vector fields, according to $(*)$ we obtain $D^c_{S_2 - S_1} X^v = 0$, and so $X^{h_1} = X^{h_2}$. $\square$

In the natural basis, the map l_h is given by

$$(l_h)_u \left(\frac{\partial}{\partial x^i}\Big|_{\pi(u)} \right) = \frac{\partial}{\partial x^i}\Big|_u - \gamma^p_{ji}(\pi(u)) y^j \frac{\partial}{\partial y^p}\Big|_u. \tag{1.5}$$

The functions $N^i_j(x, y) = \gamma^i_{kj}(x) y^k$ are called the local coefficients of the nonlinear connection N determined by the horizontal lift l_h.

2. The horizontal lift to the tangent bundle of order two

Let M be a smooth manifold of dimension n and $J_{0,p}(\mathbb{R}, M)$ the set of germs of smooth mappings $f : \mathbb{R} \to M$ with $f(0) = p$. We say that $f, g \in J_{0,p}(\mathbb{R}, M)$ are equivalent up to order k if there exists a chart (U, φ) around p such that

$$d^h_0(\varphi \circ f) = d^h_0(\varphi \circ g), \quad 1 \leq h \leq k, \tag{2.1}$$

where d denotes Frechet differentiation. It can be seen if (2.1) holds for a chart (U, φ), it holds for any other chart (V, ψ) around p.

We denote by $j^k_{0,p} f$ the equivalence class of f and set $J^k_{0,p} = \{j^k_{0,p} f, f \in J_{0,p}(\mathbb{R}, M)\}$. Then we put $T^k M = \bigcup_{p \in M} J^k_{0,p}$ and define $\pi^k : T^k M \to M$ by $\pi^k(J^k_{0,p}) = p$. $(T^k M, \pi^k, M)$ will be called the tangent bundle of order k of the manifold M. For $k = 2$, if we take $E := T^2 M$, then E is a real, smooth manifold, of dimension $3n$. For a local chart $(U, \varphi = (x^i))$ in $p \in M$ its lifted local chart in $u \in (\pi^2)^{-1}(p)$ will be denoted by $((\pi^2)^{-1}(U), \Phi = (x^i, y^{(1)i}, y^{(2)i}))$.

For each $u = (x, y^{(1)}, y^{(2)}) \in E$, the natural basis of the tangent space $T_u E$ is $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^{(1)i}}|_u, \frac{\partial}{\partial y^{(2)i}}|_u\}$.

We have two canonical submersions $\pi^2 : T^2 M \to M$ and $\pi^2_1 : T^2 M \to T^1 M \equiv TM$ which are locally expressed by: $\pi^2 : (x, y^{(1)}, y^{(2)}) \mapsto (x)$ and $\pi^2_1 : (x, y^{(1)}, y^{(2)}) \mapsto (x, y^{(1)})$, respectively.

We have two vertical distributions $V_{\alpha+1}E = \mathrm{Ker}(\pi^2_\alpha)_*$, where $(\pi^2_\alpha)_*$ is the tangent map associated to $\pi^2_\alpha, \alpha \in \{0,1\}$.

The tensor field $J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i}$ is called the 2-*almost tangent structure* on E. The 2-almost-tangent structure J has the property: 1. $J^3 = 0$, 2. $\mathrm{Im}\, J^2 = \mathrm{Ker}\, J = V_2E$. 3. $\mathrm{Im}\, J = \mathrm{Ker}\, J^2 = V_1E$. 4. $\mathrm{rank}\, J = 2n$, $\mathrm{rank}\, J^2 = n$. The vector field $\overset{2}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}}$ is called the *Liouville vector field* and is globally defined on E.

Definition 2.1 [6]. A vector field $S \in \chi(E)$ is called a semispray on E (2-semispray) if $JS = \overset{2}{\Gamma}$.

The local expression of a semispray is:

$$S = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} - 3G^i\frac{\partial}{\partial y^{(2)i}}, \tag{2.2}$$

where the functions G^i are defined on the domain of local charts.

For a vector field $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$ we denote by

$$X^{v_2} = (X^i \circ \pi^2)\frac{\partial}{\partial y^{(2)i}}$$

its vertical lift. The map $l_{v_2} : \chi(M) \to \chi(E)$, defined by $l_{v_2}(X) = X^{v_2}$ is $\mathcal{F}(M)$-linear and is called *vertical lift*, too. This means that for every $X \in \chi(M)$ and $f \in \mathcal{F}(M)$ we have $l_{v_2}(fX) = (f \circ \pi^2)l_{v_2}(X)$.

For $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$, the vector field $X^c \in \chi(E)$ given by

$$X^c = X^i\frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j}y^{(1)j}\frac{\partial}{\partial y^{(1)i}} + \left(\frac{1}{2}\frac{\partial^2 X^i}{\partial x^j \partial x^k}y^{(1)j}y^{(1)k} + \frac{\partial X^i}{\partial x^j}y^{(2)j}\right)\frac{\partial}{\partial y^{(2)i}}$$

is called the *complete lift* of the vector field X. Note that if $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$ and S is a semispray on T^2M then

$$X^c = X^i\frac{\partial}{\partial x^i} + S(X^i)\frac{\partial}{\partial y^{(1)i}} + \frac{1}{2}S^2(X^i)\frac{\partial}{\partial y^{(2)i}}.$$

Consequently, we can construct the complete lift using a semispray S and this complete lift does not depend on the choice of the semispray S.

For the vertical and complete lifts the following formulae hold:

$$J^2(X^c) = X^{v_2},$$

$$(fX)^c = \frac{1}{2}S^2(f)X^{v_2} + S(f)J(X^c) + fX^c,\ f \in \mathcal{F}(M),\ X \in \chi(M).$$

The map $X \in \chi(M) \mapsto X^c \in \chi(E)$ is not an $\mathcal{F}(M)$-linear map. Next we introduce a modification of this map so that to get an $\mathcal{F}(M)$-linear one.

Definition 2.2. An $\mathcal{F}(M)$-linear map $l_h : \chi(M) \to \chi(E)$, for which we have

$$J^2 \circ l_h = l_{v_2}, \tag{2.3}$$

is called a horizontal lift on the tangent bundle of order two.

Definition 2.3 [6]. A subbundle HE of the bundle (TE, π_E, E), which is supplementary to the vertical subbundle V_1E, that is the following Whitney sum holds

$$TE = HE \oplus V_1E, \tag{2.4}$$

is called a nonlinear connection on E.

A nonlinear connection determines an n-dimensional distribution $N : u \in TM \to N_u = H_uE \subset T_uE$.

Proposition 2.1. *Every horizontal lift l_h determines a nonlinear connection on the tangent bundle of order two.*

Proof. For every $u \in E$ we set $N_u = (l_h)_u(T_{\pi^2(u)}M)$, $\frac{\delta}{\delta x^i}|_u = (l_h)_u(\frac{\partial}{\partial x^i}|_{\pi^2(u)})$ and $\frac{\delta}{\delta y^{(1)i}}|_u = J(\frac{\delta}{\delta x^i}|_u)$. As $J^2 \circ l_h = l_{v_2}$ we obtain $J^2(\frac{\delta}{\delta x^i}) = \frac{\partial}{\partial y^{(2)i}}$. It follows that $\{\frac{\delta}{\delta x^i}|_u, \frac{\delta}{\delta y^{(1)i}}|_u, \frac{\partial}{\partial y^i}|_u\}$ are linearly independent, so they determine a basis for T_uE. Since $\{\frac{\delta}{\delta x^i}|_u\}$ is a basis for N_u, $\{\frac{\delta}{\delta y^{(1)i}}|_u = J(\frac{\delta}{\delta x^i}|_u)\}$ is a basis for $N_1(u) = J(N_u)$ and $\{\frac{\partial}{\partial y^{(2)i}}|_u\}$ is a basis for $V_2(u)$ we have $T_uE = N_u \oplus N_1(u) \oplus V_2(u)$ for each $u \in E$. But $\{\frac{\delta}{\delta y^{(1)i}}|_u, \frac{\partial}{\partial y^{(2)i}}|_u\}$ is a basis for $V_1(u)$ which means that $V_1(u) = N_1(u) \oplus V_2(u)$ and so $T_uE = N_u \oplus V_1(u)$. Hence the distribution $N : u \in E \mapsto N_u$ is supplementary to the vertical distribution V_1 and so N determines a nonlinear connection. $\square$

Conversely every nonlinear connection N determines a horizontal lift.

For a linear connection D, with the local coefficients γ^i_{jk}, we denote by D^c its complete lift given by $D^c_{X^c}Y^c = (D_XY)^c$.

The linear connection D^c preserves the vertical distributions V_1 and V_2 by parallelism.

Theorem 2.1. *Let S be a 2-semispray on E. For $X \in \chi(M)$ we define $X^{v_1}, X^h \in \chi(E)$ by:*

$$X^{v_1} = J(X^c) - D^c_S X^{v_2},$$
$$X^h = X^c - D^c_S X^{v_1} - \frac{1}{2}(D^c)^2_S X^{v_2}. \tag{2.5}$$

The maps $l_{v_1}, l_h : \chi(M) \to \chi(E)$ defined by $l_{v_1}(X) = X^{v_1}$, $l_h(X) = X^h$ are $\mathcal{F}(M)$-linear and verify $J^2 \circ l_h = l_{v_2}$, $J \circ l_{v_1} = l_{v_2}$, $J \circ l_h = l_{v_1}$. These maps are independent on the choice of the semispray S.

Proof. We proceed as in the proof of Proposition 1.1.

First we prove that (l_h) is an $\mathcal{F}(M)$-linear map. For $f \in \mathcal{F}(M)$ we have
$$(fX)^h = (fX)^c - D^c_S(fX)^{v_1} - \frac{1}{2}(D^c)^2_S(fX)^{v_2}.$$
Since $(fX)^c = fX^c + S(f)J(X^c) + \frac{1}{2}S^2(f)X^{v_2}$, $(fX)^{h_1} = fX^{h_1}$ and $(fX)^{v_2} = fX^{v_2}$ we obtain $(fX)^h = fX^c + S(f)J(X^c) + \frac{1}{2}S^2(f)X^{v_2} - S(f)(J(X^c) - D^c_S X^{v_2}) - \frac{1}{2}(S^2(f)X^{v_2} + 2S(f)D^c_S X^{v_2} + f(D^c)^2_S X^{v_2}) = fX^h$.

For every $X \in \chi(M)$ we have $J^2(X^c) = X^{v_2}$. Since $\operatorname{Ker} J^2 = V_1$, $D^c_S X^{h_1}$ and $(D^c)^2_S X^{v_2}$ are vertical vector fields, we obtain $JD^c_S X^{h_1} = 0$ and $J^2((D^c)^2_S X^{v_2}) = 0$. It results $(J^2 \circ l_h)(X) = J^2(X^c) = X^{v_2} = l_{v_2}(X)$, $\forall X \in \chi(M)$. In this way we obtain that the map l_h is a *horizontal lift*. Next we prove that this map depends on the linear connection D on M, only. Let S and $\widetilde{S}$ two semisprays and

$$X^{v_1} = J(X^c) - D^c_S X^{v_2}, \qquad X^h = X^c - D^c_S X^{v_1} - \frac{1}{2}(D^c)^2_S X^{v_2};$$
$$\widetilde{X}^{v_1} = J(X^c) - D^c_{\widetilde{S}} X^{v_2}, \qquad \widetilde{X}^h = X^c - D^c_{\widetilde{S}} \widetilde{X}^{v_1} - \frac{1}{2}(D^c)^2_{\widetilde{S}} X^{v_2}.$$

the horizontal lifts of a vector field $X \in \chi(M)$ constructed with the semisprays S and $\widetilde{S}$, respectively.

We have $X^{v_1} - \widetilde{X}^{v_1} = -D^c_{S-\widetilde{S}} X^{v_2}$. From the definition of D^c we have $D^c_X Y^{v_2} =$ for every $X \in \chi^{V_2}(E)$ and $Y \in \chi(M)$. As $S - \widetilde{S}$ belongs to $\chi^{V_2}(E)$, we obtain $D^c_{S-\widetilde{S}} X^{v_2} = 0$, and so $X^{v_1} = \widetilde{X}^{v_1}$. Using the definition of X^h and $\widetilde{X}^h$ we obtain: $X^h - \widetilde{X}^h = -D^c_{S-\widetilde{S}} X^{v_1} - \frac{1}{2}(D^c)^2_{S-\widetilde{S}} X^{v_2}$. But for $D^c_{S-\widetilde{S}} X^{v_1} = (G^i - \widetilde{G}^i) D^c_{\frac{\partial}{\partial y^{(2)i}}} X^{v_1}$ and $X^{v_1} \in \chi^{V_1}(E)$ we obtain $D^c_{\frac{\partial}{\partial y^{(2)i}}} X^{v_1} = 0$ that means $D^c_{S-\widetilde{S}} X^{v_1} = 0$. In a similar way we get $D^c_{\frac{\partial}{\partial y^{(2)i}}} X^{v_2} = 0$ and also we have $D^c_{S-\widetilde{S}} X^{v_2} = 0$ and $(D^c)^2_{S-\widetilde{S}} X^{v_2} = 0$. We proved that $X^h = \widetilde{X}^h$ and so the horizontal lift X^h of a vector field $X \in \chi(M)$ to the tangent bundle of order two is independent on the choice of the semispray S. □

Proposition 2.2. *In the natural basis, the maps l_h and l_{v_1} have the following expressions:*

$$
\begin{aligned}
l_{v_1}\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N_i^j} \frac{\partial}{\partial y^{(2)j}} \\
l_h\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial}{\partial x^i} - \underset{(1)}{N_i^j} \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N_i^j} \frac{\partial}{\partial y^{(2)j}},
\end{aligned} \tag{2. 6}
$$

where

$$
\begin{aligned}
\underset{(1)}{N_i^j} &= \gamma^j_{pi} y^{(1)p}, \\
\underset{(2)}{N_i^j} &= \frac{1}{2}\left(\frac{\partial \gamma^j_{ip}}{\partial x^k} - \gamma^j_{mp}\gamma^m_{ik}\right) y^{(1)p} y^{(1)k} + \gamma^j_{ip} y^{(2)p}.
\end{aligned} \tag{2.7}
$$

Proof. Taking into account (2.5) and $(\frac{\partial}{\partial x^i})^{v_1} = \frac{\partial}{\partial y^{(1)i}} - D^c_S \frac{\partial}{\partial y^{(2)i}}$ we obtain

$$\left(\frac{\partial}{\partial x^i}\right)^{v_1} = \frac{\partial}{\partial y^{(1)i}} - \gamma^k_{ij} y^{(1)j} \frac{\partial}{\partial y^{(2)k}}.$$

Using the notation $\underset{(1)}{N_i^j} = \gamma^j_{pi} y^{(1)p}$ we see that the first formula (2.6) holds. Next we denote

$$\frac{\delta}{\delta y^{(1)i}} = l_{v_1}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}} - \gamma^k_{ij} y^{(1)j} \frac{\partial}{\partial y^{(2)k}}.$$

For the horizontal lift l_h we have

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - D^c_S \frac{\delta}{\delta y^{(1)i}} - \frac{1}{2}(D^c)^2_S \frac{\partial}{\partial y^{(2)i}}.$$

As $D^c_S \frac{\partial}{\partial y^{(2)i}} = \gamma^k_{ij} y^{(1)j} \frac{\partial}{\partial y^{(2)k}} = \underset{(1)}{N}{}^k_i \frac{\partial}{\partial y^{(2)k}}$ it results:
$(D^c)^2_S \frac{\partial}{\partial y^{(2)i}} = D^c_S \underset{(1)}{N}{}^k_i \frac{\partial}{\partial y^{(2)k}}$. Accordingly we have

$$\begin{aligned} l_h\left(\frac{\partial}{\partial x^i}\right) = {} & \frac{\partial}{\partial x^i} - \gamma^j_{ik} y^{(1)k} \frac{\partial}{\partial y^{(1)j}} \\ & - \frac{1}{2}\left\{\left(\frac{\partial \gamma^j_{ip}}{\partial x^k} - \gamma^j_{mp}\gamma^m_{ik}\right) y^{(1)p} y^{(1)k} + 2\gamma^j_{ip} y^{(2)p}\right\} \frac{\partial}{\partial y^{(2)j}}. \end{aligned}$$

The horizontal lift determines, by the Proposition 2.1 a nonlinear connection N. This connection is just Miron's nonlinear connection, [6].

We remark that the map l_{v_1} determines a distribution N_1 which is supplementary to the vertical distribution V_2 in the distribution V_1 i.e. $V_1(u) = N_1(u) \oplus V_2(u), \forall u \in E$. Also, because of $J \circ l_h = l_{v_1}$ we have $J(N) = N_1$. □

3. Horizontal lift to the tangent bundle of higher order

The problems which were presented in the previous section can be extended to the general case of order $k > 2$. In this section we point out only the differences from $k = 2$ case.

Let $(E := T^k M, \pi^k, M)$ be the tangent bundle of order k of a real, smooth, n-dimensional manifold M.

For a local chart $(U, \phi = (x^i))$ on M we denote by $\big((\pi^k)^{-1}(U), \Phi = (x^i, y^{(1)i}, y^{(2)i}, \dots, y^{(k)i})\big)$ its induced local chart on $T^k M$.

For every $u = (x, y^{(1)}, y^{(2)}, \dots, y^{(k)}) \in E$, the natural basis of the tangent space $T_u E$ will be denoted by $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^{(1)i}}|_u, \dots, \frac{\partial}{\partial y^{(k)i}}|_u\}$.

We consider $V_k(u) \subset \dots \subset V_1(u)$ the vertical distributions induced by the natural submersions $\pi^k_{k-1}, \dots, \pi^k_1, \pi^k$. *The k-almost tangent structure of the tangent bundle of order* is a tensor field of $(1,1)$-type, which is locally expressed by

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \dots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i}. \tag{3.1}$$

Definition 3.1 [6]. A vector field $S \in \chi(E)$ is said to be a semispray on E (k-semispray) if $JS = \overset{k}{\Gamma}$, where

$$\overset{k}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k)i}}$$

is the Liouville vector field.

The local expression of a k-semispray is given by

$$S = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i\frac{\partial}{\partial y^{(k)i}}, \tag{3.2}$$

the functions G^i being defined on the domain of a local chart.

For a vector field $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$ we denote by

$$X^{v_k} = (X^i \circ \pi^k)\frac{\partial}{\partial y^{(k)i}}$$

its vertical lift. The map $l_{v_k} : \chi(M) \to \chi(E)$, which is defined by $l_{v_k}(X) = X^{v_k}$ is $\mathcal{F}(M)$-linear and is called also *the vertical lift*. This means that for every $X \in \chi(M)$ and $f \in \mathcal{F}(M)$ we have $l_{v_k}(fX) = (f \circ \pi^k)l_{v_k}(X)$. For $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$ and S a k-semispray, the vector field $X^c \in \chi(E)$ defined by:

$$X^c = X^i\frac{\partial}{\partial x^i} + \frac{1}{1!}S(X^i)\frac{\partial}{\partial y^{(1)i}} + \frac{1}{2!}S^2(X^i)\frac{\partial}{\partial y^{(2)i}} + \cdots + \frac{1}{k!}S^k(X^i)\frac{\partial}{\partial y^{(k)i}}$$

is called *the complete lift* of the vector field X.

A direct consequence of $X^i = X^i(x)$ is $S^\alpha(X^i) = \widetilde{S}^\alpha(X^i)$ for every two semisprays S and $\widetilde{S}$ and so the complete lift of a vector field X is independent on the choice of the semispray S.

For the vertical and complete lifts the following formulae hold:

$$\begin{aligned} &J^k(X^c) = X^{v_k}, \\ &(fX)^c = \sum_{\alpha=0}^{k}\frac{1}{\alpha!}S^\alpha(f)J^\alpha(X^c), \quad f \in \mathcal{F}(M),\ X \in \chi(M). \end{aligned} \tag{3.3}$$

It can be seen from the second formula (3.3) that the map $X \in \chi(M) \mapsto X^c \in \chi(E)$ is not an $\mathcal{F}(M)$-linear map. Next, we modify this map such that the new map will be $\mathcal{F}(M)$-linear.

Definition 3.2. An $\mathcal{F}(M)$-linear map $l_h : \chi(M) \to \chi(E)$, for which we have

$$J^k \circ l_h = l_{v_k}, \tag{3.4}$$

is called a horizontal lift on the tangent bundle of order k.

Definition 3.3 [6]. A subbundle HE of the tangent bundle (TE, π_E, E), which is supplementary to the vertical subbundle V_1E, i.e. the following Whitney sum holds

$$TE = HE \oplus V_1E, \tag{3.5}$$

is called a nonlinear connection.

A nonlinear connection determines a horizontal n-dimensional distribution $N : u \in TM \to N_u = H_uE \subset T_uE$.

Like in the $k = 1$ or $k = 2$ cases we have that every horizontal lift l_h determines a nonlinear connection on the tangent bundle of order k. Conversely every nonlinear connection N determines a horizontal lift.

Let D be a linear connection on M with the local coefficients γ^i_{jk}. We denote by D^c its complete lift. This is uniquely determined by

$$D^c_{X^c} Y^c = (D_X Y)^c. \tag{3.6}$$

For this linear connection we have also the next formulae

$$D^c_{J^\alpha(X^c)} Y^c = D^c_{X^c} J^\alpha (Y^c) = J^\alpha ((D_X Y)^c), \quad \alpha = \overline{0, k}. \tag{3.7}$$

Theorem 3.1. *Let S be a semispray on E. For $X \in \chi(M)$ we define $X^{v_{k-1}}, \ldots, X^{v_1}, X^h \in \chi(E)$ by:*

$$\begin{aligned} X^{v_{k-1}} &= J^{k-1}(X^c) - \frac{1}{1!} D^c_S (X^{v_k}), \\ X^{v_{k-2}} &= J^{k-2}(X^c) - \frac{1}{1!} D^c_S X^{v_{k-1}} - \frac{1}{2!} (D^c)^2_S X^{v_k}, \ldots, \\ X^h &= X^c - \frac{1}{1!} D^c_S X^{v_1} - \frac{1}{2!} (D^c)^2_S X^{v_2} - \cdots - \frac{1}{k!} (D^c)^k_S X^{v_k}. \end{aligned} \tag{3.8}$$

The maps $l_{v_\alpha}, l_h : \chi(M) \to \chi(E)$, $\alpha = 1, 2, \dots, k$ *defined by* $l_{v_\alpha}(X) = X^{v_\alpha}$, $l_h(X) = X^h$ *are* $\mathcal{F}(M)$*-linear and verify* $J^k \circ l_h = l_{v_k}$, $J^\alpha \circ l_h = l_{v_\alpha}$. *These maps are independent on the choice of the semispray* S.

Proof. For the maps $l_{v_{k-1}}$ and $l_{v_{k-2}}$ the stated properties are proved as in Proposition 1.1 and Theorem 2.1. We assume that the stated properties are true for $l_{v_{k-\beta}}$ for $\forall \beta \in 1, 2, \dots, \alpha - 1$ with $1 \leq \alpha \leq k$ and we prove, using (3.3) that l_{v_α} verifies also the required properties. □

We denote by N the nonlinear connection induced by the horizontal lift determined in the above. Let $N_1 = J(N)$, $N_2 = J^2(N), \dots, N_{k-1} = J^{k-1}(N)$.

We set $\frac{\delta}{\delta x^i} = l_h(\frac{\partial}{\partial x^i})$ and $\frac{\delta}{\delta y^{(\alpha)i}} = l_{v_\alpha}(\frac{\partial}{\partial x^i})$, $\alpha \in \{1, \dots, k-1\}$. Since $J^\alpha \circ l_h = l_{v_\alpha}$ we obtain $J^\alpha(\frac{\delta}{\delta x^i}) = \frac{\delta}{\delta y^{(\alpha)i}}$. On this way we get for every $u \in E$ $\{\frac{\delta}{\delta x^i}|_u, \frac{\delta}{\delta y^{(1)i}}|_u, \dots, \frac{\delta}{\delta y^{(k-1)i}}|_u, \frac{\partial}{\partial y^{(k)i}}|_u\}$ a basis for T_uE which is adapted to the direct decomposition

$$T_uE = N(u) \oplus N_1(u) \oplus \cdots \oplus N_{k-1}(u) \oplus V_k(u).$$

References

[1] M. Anastasiei, On Deflection Tensor field in Generalized Lagrange Spaces, Lagrange and Finsler geometry, *Kluwer Acad. Publ.*, FTPH no. 76, 1996, 1–14.

[2] M. Anastasiei and I. Bucataru, A notable Submersion in the Higher Order Geometry, Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, *Geometry Balkan Press*, 1997, 1–10.

[3] I. Bucataru, Characterization of the nonlinear connection in the higher order geometry, *Balkan Journal of Geometry and its Applications,* 2 **2** (1997), 13–22.

[4] V. Cruceanu, Objets géometériques h-invariants sur le fibré tangent, *Atti Accad. Naz. Lincei Rend. Cl. Mat. Natur., Serie VIII, LVII* **6** (1974), 548–558.

[5] V. Cruceanu, On h-invariant vector fields in a vector bundle, *Rev. Roumaine Math. Pures Appl.* **42**, 9–10 (1997), 759–766.

[6] R. Miron, The Geometry of Higher Order Lagrange Spaces, Applications to Mechanics and Physics, *Kluwer*, *Dordrecht*, FTPH no. 82, 1997.

[7] R. Miron and M. Anastasiei, The geometry of Lagrange spaces: Theory and Applications, *Kluwer*, *Dordrecht*, FTPH no. 59, 1994.

[8] K. Yano and S. Ishihara, Tangent and cotangent bundles, *Marcel Dekker Inc.*, *New York*, 1973.

IOAN BUCATARU
"AL. I. CUZA" UNIVERSITY IASI
FACULTY OF MATHEMATICS
B–DUL COPOU, NO. 11
6600 IAŞI
ROMANIA

E-mail: bucataru@uaic.ro

(Received December 22, 1997; revised March 4, 1999)