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Quasi-additive and quasi-multiplicative functions with regularity properties

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Abstract. In 1946, P. ERDŐS [3] proved the remarkable theorem that if f is an additive monotonic function, then it is a constant multiple of log. He also posed the conjecture that if it is additive and $\frac{1}{x}\sum_{n\le x}|f(n+1)-f(n)| \to 0$ $(x\to\infty)$, then f is a constant multiple of log. While several authors subsequently gave alternate (and usually simpler) proofs of the Erdős theorem it was KÁTAI [4] and WIRSING [5] who independently proved the Erdős conjecture. We here introduce the concept of quasi-additive functions (analogous to the quasi-multiplicative functions introduced by the second author in 1985) and prove that the above results hold for quasi-additive functions, besides establishing several other theorems.

1. Introduction

In 1985, SUBBARAO introduced weakly multiplicative arithmetic functions $f(n)$ (later renamed quasi-multiplicative arithmetic functions) as those for which the property

$$f(np) = f(n)f(p) \tag{1.1}$$

holds for all primes p and natural numbers n which are relatively prime to p. In that abstract Subbarao also announced the theorem that if $f(n)$ is quasi-multiplicative, integer valued and satisfies

$$f(n+p) \equiv f(n) \pmod{p} \tag{1.2}$$

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for all natural numbers n and all primes p, then $f(n) \equiv n^c$ for some constant $c \geq 0$, or else $f(n)$ is identically zero.

Later, in the paper [2] J. Fabrykowski and M. V. Subbarao gave a proof of the above theorem and made the conjecture that the theorem continues to hold even if the relation (1.2) holds only for an infinity of primes instead of for all primes. This conjecture is still open.

In the present paper we introduce the notion of quasi-additive functions and obtain several results involving quasi-additive and quasi-multiplicative functions with regularity properties.

An arithmetic function f is called quasi-additive if for all primes p and natural numbers n coprime to p the relation

$$f(np) = f(n) + f(p) \tag{1.3}$$

holds.

Let $\mathcal{M}$ be the set of square-free numbers, and $\mathcal{K}$ be the set of square-full numbers. It is clear that each natural number n can be written uniquely as $n = Km$, where $(K, m) = 1$ and $K \in \mathcal{K}$, $m \in \mathcal{M}$. We can see immediately that f is quasi-additive if and only if for every integer n, $f(m) = f(K) + f(n)$, and f is additive on the set $\mathcal{M}$. A similar assertion is valid for quasi-multiplicative functions.

2. Theorems concerning quasi-additive and quasi-multiplicative functions

In 1946 P. Erdős [3] proved that if f is an additive function for which either $f(n+1) - f(n) \to 0$, or $f(n) \leq f(n+1)$ $(n = 1, 2, \dots)$, then $f(n)$ is a constant multiple of $\log n$. We shall prove that the same assertions hold for quasi-additive functions.

Theorem 1. *If a quasi-additive function f is monotonic, then it is a constant multiple of* log.

Theorem 2. *If f is quasi-additive and*

$$\frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| \to 0 \quad (x \to \infty), \tag{2.1}$$

then f is a constant multiple of log.

The above theorem for additive functions was formulated as a conjecture by Erdős [3], and proved by Kátai [4], and Wirsing [5] independently.

The multiplicative function $g(n) = n^s$, $s \in \mathbb{C}$ behaves quite regularly as $n \to \infty$. Especially, $(n+1)^s - n^s \to 0$ if $\mathrm{Res} < 1$. Giving a positive answer to a conjecture of Kátai, in 1984 E. Wirsing proved, that if g is a complex-valued multiplicative function and $|g(n)| = 1$ $(n = 1, 2, \dots)$, such that $|g(n+1) - g(n)| \to 0$, then $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

The proof appeared only in 1996 in printed form [8].

From Wirsing's theorem we can obtain that if a complex-valued multiplicative function g satisfies $\Delta g(n) := g(n+1) - g(n) \to 0$ $(n \to \infty)$, then either $g(n) \to 0$ or $g(n) = n^s$ $(\mathrm{Res} < 1)$ (see [6]).

We shall prove

Theorem 3. *If g is a quasi-multiplicative function such that $|g(n)| = 1$, $\Delta g(n) \to 0$, then g is multiplicative. As a consequence of Wirsing's theorem, $g(n) = n^{i\tau}$.*

At present we cannot characterize the class of those multiplicative functions g for which

$$\frac{1}{x} \sum_{n \le x} |g(n+1) - g(n)| \to 0 \quad (x \to \infty) \tag{2.2}$$

holds. L. Murata and J. L. Mauclaire [7] proved that if (2.2) holds, $|g(n)| = 1$ $(n = 1, 2, \dots)$, and g is multiplicative, then it is completely multiplicative.

Now we shall prove

Theorem 4. *Let g be quasi-multiplicative, $|g(n)| = 1$ $(n = 1, 2, \dots)$. Assume that (2.2) holds. Then g is completely multiplicative.*

3. Lemmata

Lemma 1. *Let $1 \le Y_0$, $x \ge e^{Y_0}$, and let h be an arbitrary additive function defined on the set $\mathcal{M}$,*

$$a(x) = \sum_{Y_0 \le p \le x} \frac{h(p)}{p}, \qquad b_1^2(x) = \sum_{Y_0 \le p \le x} \frac{h^2(p)}{p},$$

$$b_2^2(x) = \sum_{x^{1/4} < p \le x} \frac{h^2(p)}{p}, \quad \Delta(x) = \prod_{p < x} \left(1 - \frac{1}{p}\right).$$

Then, there exists an absolute constant c_1 such that

$$\sum (h(m) - a(x))^2 \le c_1 x \Delta(Y_0) b_1^2(x) + c_1 x b_2^2(x),$$

where on the right hand side m runs over those square-free numbers up to x the smallest prime factor of which are larger than Y_0.

Remark. This Turán–Kubilius type inequality can be proved in a routine way by using the Eratosthenian sieve. We omit the proof.

Lemma 2 (P. Erdős [3]). *Assume that f is additive and there are positive constants c_1, c_2 and a sequence $x_\nu \to \infty$ $(\nu \to \infty)$, such that for all ν we can choose suitable integers $1 \le a_1 < a_2 < \cdots < a_t \le x_\nu$ such that $t > c_1 x_\nu$ and*

$$|f(a_j) - f(a_k)| \le c_2$$

for every $j, k \le t$.

Then f is finitely distributed, i.e., $f(n) = c \log n + t(n)$, where

$$\sum_p \frac{\min(1, t^2(p))}{p} < \infty.$$

4. Proof of Theorem 1

Assume that $f(n) \le f(n+1)$ $(n = 1, 2, \dots)$. If n is an odd square-free number, then $f(n) \le f(\nu) \le f(2n) = f(2) + f(n)$ for every $\nu \in \mathcal{M}$ from the interval $\nu \in (n, 2n)$, thus for at least cn integers $\nu \in \mathcal{M}$. Let $F(n)$ be the additive function for which $F(K) = 0$ if $K \in \mathcal{K}$, and $F(m) = f(m)$ if $m \in \mathcal{M}$. Then F is a finitely distributed additive function, consequently by Lemma 2,

$$F(n) = c \log n + t(n), \tag{4.1}$$

and

$$\sum_p \frac{\min(1, t^2(p))}{p} < \infty. \tag{4.2}$$

Let

$$\mathcal{R} = \{p \mid |t(p)| \ge 1\}.$$

From (4.2) we obtain that

$$\sum_{p\in\mathcal{R}} \frac{1}{p} < \infty. \tag{4.3}$$

Let $\mathcal{I}$ be the set of those integers m for which:

(a) $m \in \mathcal{M}$,

(b) the smallest prime factor is larger than Y_0,

(c) they do not have prime divisors from the set $\mathcal{R}$.

Let $T(x) = \#\{n \le x \mid n \in \mathcal{I}\}$, $s(n) = 1$ or 0 according as to n does or does not belong to $\mathcal{I}$.

Then, by the Eratosthenian sieve we obtain that

$$T(x) = (1 + o(1))x \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{h(p^\alpha)}{p^\alpha}\right), \tag{4.4}$$

where

$$h(p^\alpha) = s(p^\alpha) - s(p^{\alpha-1}), \quad \text{and}$$

$$s(p) = 1 \quad \text{if } p \notin \mathcal{R},\ p > Y_0,$$

$$s(p) = 0 \quad \text{if either } p \in \mathcal{R}, \text{ or } p \le Y_0.$$

Furthermore,

$$s(p^\alpha) = 0 \quad \text{if } \alpha \ge 2.$$

Consequently, from (4.4) we immediately deduce that

$$T(x) = c(Y_0)(1 + o(1))x, \tag{4.5}$$

where

$$c(Y_0) = \prod_{p\le Y_0} \left(1 - \frac{1}{p}\right) \prod_{\substack{p\in\mathcal{R}\\ p>Y_0}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p\notin\mathcal{R}\\ p>Y_0}} \left(1 - \frac{1}{p^2}\right). \tag{4.6}$$

Let N and M be arbitrary positive integers. Let furthermore ε_1, ε_2, δ be arbitrarily small positive numbers.

Let us choose Y_0 so that $Y_0 > \max(N, M)$,

$$c(Y_0) \geq (1-\varepsilon_2)\Delta(Y_0), \tag{4.7}$$

and

$$c_1 \sum_{\substack{Y_0<p \\ p\notin\mathcal{R}}} \frac{t^2(p)}{p} < \varepsilon_1^4, \tag{4.8}$$

and let X_0 be so that for each $X \geq X_0$

$$c_1 \sum_{\substack{X^{1/4}\leq p\leq x \\ p\notin\mathcal{R}}} \frac{t^2(p)}{p} < \varepsilon_1^4\Delta(Y_0). \tag{4.9}$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the set of integers from the interval $\left(\frac{x(1-\delta)}{N}, \frac{x}{N}\right)$, and from $\left(\frac{x}{M}, \frac{x(1+\delta)}{M}\right)$, respectively.

Let ν and μ run over $\mathcal{L}_1 \cap \mathcal{I}$ and $\mathcal{L} \cap \mathcal{I}$, respectively.

Assume now that x is a large value.

From Lemma 1 we obtain that with the exception of at most $\varepsilon T(\frac{x}{N})$ values of ν, and at most $\varepsilon T(\frac{x}{M})$ values of μ,

$$t(\nu) - \tilde{a}\left(\frac{x}{N}\right) \in [-\varepsilon, \varepsilon], \quad t(\mu) - \tilde{a}\left(\frac{x}{M}\right) \in [-\varepsilon, \varepsilon] \tag{4.10}$$

hold, if x is large enough. Here

$$\tilde{a}(x) = \sum_{\substack{Y_0<p<x \\ p\notin\mathcal{R}}} \frac{t(p)}{p}. \tag{4.11}$$

Indeed, apply Lemma 1 for the additive function h the values on primes of which are

$$h(p) = \begin{cases} t(p) & \text{if } p \notin \mathcal{R} \\ 0 & \text{if } p \in \mathcal{R}. \end{cases}$$

Thus, by (4.8) and (4.9)

$$\sum_{\nu\in\mathcal{L}_1} \left(t(\nu) - \tilde{a}\left(\frac{x}{N}\right)\right)^2 \leq 2\varepsilon_1^4\Delta(Y_0)\frac{x}{N},$$

and consequently

$$\#\{\nu \in \mathcal{L}_1,\ |t(\nu) - \tilde{a}(x/N)| \geq \varepsilon\} \leq 2\varepsilon_1^2 \Delta(Y_0)\frac{x}{N} < \varepsilon T\left(\frac{x}{N}\right).$$

Similarly, we can deduce that

$$\#\{\mu \in \mathcal{L}_2,\ |t(\mu) - \tilde{a}(x/M)| \geq \varepsilon\} < \varepsilon T\left(\frac{x}{M}\right).$$

Let us observe furthermore that $\tilde{a}\left(\frac{x}{N}\right) = \tilde{a}\left(\frac{x}{M}\right) + o(1)$ as $x \to \infty$.

Thus, if x is large, then we can always choose a pair $\nu^* \in \mathcal{L}_1$, $\mu^* \in \mathcal{L}_2$ for which $t(\mu^*) - t(\nu^*) \in [-2\varepsilon, 2\varepsilon]$. Since $\nu^* N < \mu^* M$, we have $f(\nu^* N) \leq f(\mu^* M)$, consequently

$$\begin{aligned} f(N) - F(M) &\leq F(\mu^*) - F(\nu^*) = c\log\frac{\mu^*}{\nu^*} + t(\mu^*) - t(\nu^*) \\ &\leq c\log\frac{x(1+\delta)N}{Mx(1-\delta)} + 2\varepsilon = c\log\frac{N}{M} + c\log\frac{1+\delta}{1-\delta} + 2\varepsilon. \end{aligned}$$

Since ε and δ are arbitrary, we get $f(N) - f(M) \leq c\log\frac{N}{M}$. Interchanging the values N and M, we immediately get that $f(N) - f(M) = c\log\frac{N}{M}$, and so that $f(N) = c\log N$.

The proof of the theorem is complete.

Remark. The same method gives the following

Theorem 1a. *Let f be an additive monotonic function on $\mathcal{M}$. Then $f(n) = c\log n$.*

5. Proof of Theorem 2

It is enough to prove that for each coprime pair K_1, K_2 and $K = K_1K_2$ we have:

$$f(K) = f(K_1) + f(K_2).$$

Let $\Delta_j f(n) := f(n+j) - f(n)$. From (2.1) we have that

$$\frac{1}{x}\sum_{n\leq x}\max_{1\leq j\leq K}|\Delta_j f(n)| \to 0 \quad (x \to \infty). \tag{5.1}$$

Let $\mathcal{H}$ denote the set of those square-free numbers m for which

(1) mK_2+1 is square-free and coprime to K_1,

(2) $(m,K)=1$.

If $m\in\mathcal{H}$, then

$$\begin{aligned} f(Km+K_1)-f(Km) &= f(K_1)+f(K_2m+1)-f(K)-f(m)\\ &= (f(K_1)+f(K_2)-f(K))+(f(K_2m+1)-f(K_2m)), \end{aligned}$$

whence

$$\begin{aligned} &|f(K_1)+f(K_2)-f(K)|\sum_{\substack{m\in\mathcal{H}\\ m\le x}}1\\ &\le \sum_{m\le x}|\Delta_{K_1}f(Km)|+\sum_{m\le x}|\Delta f(K_2m)|. \end{aligned} \tag{5.2}$$

Let now K be an arbitrary integer, with prime decomposition $K=p_1^{\alpha_1}\dots p_r^{\alpha_r}$, $r\ge 2$. Let $K_1=p_1^{\alpha_1}, p_1$ be odd. Then $K_2=p_2^{\alpha_2}\dots p_r^{\alpha_r}$. Let $a=1$ if $p_1\nmid K_2+1$ and $a=-1$, if $p_1|K_2+1$. In the last case $p_1\nmid K_2-1$. Let ν run over the arithmetic progression $\equiv a \pmod K$. Observe that for $\nu=a+tK$, $(\nu K_2+1, K_1)=aK_2+1$, $p_1=1$, $(\nu,K)=1$. The density of the square-free numbers $\equiv a \pmod K$ is clearly positive, so from (5.2) we obtain that

$$f(K_1)+f(K_2)=f(K). \tag{5.3}$$

Repeating this argument we get

$$f(p_1^{\alpha_1}\dots p_r^{\alpha_r})=f(p_1^{\alpha_1})+\dots+f(p_r^{\alpha_r}).$$

Thus f is additive, and for additive functions the assertion is known. The proof is complete.

Remark. The referee noted that Theorem 2 can be derived from our Theorem 4 directly in the following way: if f is a quasi-additive function satisfying (2.1), then for a real number t the quasi-multiplicative function $\exp(itf)$ satisfies (2.2). Since its modulus is 1, by Theorem 4 it is a multiplicative function, and since this holds for all t, one gets immediately that f is additive.

6. Proof of Theorem 4

Let K, K_1, K_2 and $\mathcal{H}$ be defined as in Section 5. Since

$$\begin{aligned} g(Km+K_1)\bar{g}(Km) &= g(K_1)\bar{g}(K)g(K_2m+1)\bar{g}(m) \\ &= g(K_1)g(K_2)\bar{g}(K)g(K_2m+1)\bar{g}(K_2m), \end{aligned}$$

we have

$$g(K_1)g(K_2)\bar{g}(K)-1 = \xi^{(m)}\eta_0^{(m)}\dots\eta_{K_1-1}^{(m)}-1, \tag{6.1}$$

where

$$\xi^{(m)} = \frac{g(K_2m)}{g(K_2m+1)}, \quad \eta_j^{(m)} = \frac{g(Km+j+1)}{g(Km+j)} \qquad (j=0,\dots,K_1-1).$$

The absolute value of the right hand side is less than

$$c\left\{|\xi^{(m)}-1|+|\eta_0^{(m)}-1|+\dots+|\eta_{K_1-1}^{(m)}-1|\right\},$$

where c is a suitable constant that may depend only on K_1.

Hence, by (2.2) we deduce that

$$\left|\frac{g(K_1)g(K_2)}{g(K)}-1\right|\sum_{\substack{m\le x\\ m\in\mathcal{H}}}1 = o(x) \quad (x\to\infty).$$

Repeating the argument which we used in the proof of Theorem 2, we deduce that

$$g(p_1^{\alpha_1}\dots p_r^{\alpha_r}) = g(p_1^{\alpha_1})g(p_2^{\alpha_2}\dots p_r^{\alpha_r})$$

if $r\ge 2$ and p_1 odd. Thus g is multiplicative, and by the theorem of Murata and Mauclaire, g is completely multiplicative.

7. Proof of Theorem 3

This is an immediate consequence of Theorem 4.

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