

A note on additive commutativity-preserving mappings

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Abstract. We characterize additive surjective commutativity-preserving mappings on M_n , $n \geq 2$.

The problem of characterizing linear transformations on M_n , the algebra of $n \times n$ complex matrices, that preserve some properties, has been considered in a number of papers. It turns out that this kind of mapping is often of the form

$$(1) \quad X \mapsto \sigma AXA^{-1} + f(X)I \quad \text{or} \quad X \mapsto \sigma AX^{\text{tr}}A^{-1} + f(X)I,$$

where σ is a non-zero complex number, X^{tr} denotes the transpose of X , and f is a linear functional on M_n . It is natural to try to get similar results studying not linear but merely additive preservers. OMLADIČ and ŠEMRL [10], [9] characterized additive spectrum-preserving mappings and additive mappings preserving operators of rank one. We say that ϕ *preserves commutativity* if $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$ (briefly $A \leftrightarrow B$), and it *preserves commutativity in both directions* if also $\phi(A) \leftrightarrow \phi(B)$ implies $A \leftrightarrow B$. Bijective additive mappings preserving commutativity on more general algebras have been described by BREŠAR, MIERS, BANNING and MATHIEU [4], [5], [2]. This note is a continuation of the work of the present author [11], where we obtained the general form of an additive surjective mapping on M_n , $n \geq 3$, that preserves commutativity in both directions. The methods we use here are different, and we

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replace the assumption of preserving commutativity in both directions by the weaker one, preserving commutativity in one direction only. Moreover, the characterization of such mappings on M_2 is included. For $n \geq 3$ we obtain as a result the mappings of the form

$$(2) \quad X \mapsto \sigma T(X) + p(X)I$$

where $\sigma \neq 0$ is a complex constant, p a complex valued additive mapping on M_n , and $T : M_n \rightarrow M_n$ is defined either by $[x_{ij}] \mapsto A[f(x_{ij})]A^{-1}$, or $[x_{ij}] \mapsto A[f(x_{ij})]^{\text{tr}}A^{-1}$ for some invertible matrix A and a ring automorphism f on $\mathbb{C}$. The mapping $\lambda \mapsto \bar{\lambda}$ of a complex number to its conjugate is a nontrivial continuous ring automorphism of $\mathbb{C}$. Moreover, there exist nowhere continuous ring automorphisms of $\mathbb{C}$ [1]. It is not surprising that the result for $n = 2$ differs essentially from that for $n \geq 3$. Even in the linear case the mappings of the form (1) are not the only ones that arise as bijective commutativity preservers on M_2 [13]. Any mapping of the form (2) can be regarded as a compositum of a linear bijective commutativity-preserving mapping and a ring automorphism $[x_{ij}] \mapsto [f(x_{ij})]$, additively perturbed by a mapping $X \mapsto p(X)I$. The same holds true in the two dimensional case. The set of all bijective linear mappings $\phi : M_2 \rightarrow M_2$ satisfying $\phi(I) = \lambda I$, for some $\lambda \neq 0$, is equal to the set of all bijective linear mappings on M_2 that preserve commutativity. This is a straightforward consequence of the fact that the commutant X' (the set of all matrices from M_n commuting with X) of any non-scalar matrix $X \in M_2$ is only two dimensional, i.e.:

$$(3) \quad X' = \{\alpha X + \beta I, \alpha, \beta \in \mathbb{C}\}.$$

Before giving the proofs we introduce some notation: $[A, B] = AB - BA$, $E_{ij} = [\delta_{ij}]$, where δ_{ij} is the Kronecker symbol. The mapping $\phi : M_n \rightarrow M_n$ is called f -quasilinear, for some $f : \mathbb{C} \rightarrow \mathbb{C}$, if it is additive, and if the relation $\phi(\alpha X) = f(\alpha)\phi(X)$ holds for all complex numbers α and $X \in M_n$.

Theorem. *Let ϕ be an additive surjective commutativity-preserving mapping on M_n , $n \geq 2$.*

If $n \geq 3$ then there exists a ring automorphism $f : \mathbb{C} \rightarrow \mathbb{C}$, a non-zero complex constant σ , an invertible matrix A and an additive function $p : M_n \rightarrow \mathbb{C}$ such that ϕ is either of the form

$$(a) \quad \phi([x_{ij}]) = \sigma A[f(x_{ij})]A^{-1} + p([x_{ij}])I, \quad [x_{ij}] \in M_n,$$

or

$$(b) \quad \phi([x_{ij}]) = \sigma A[f(x_{ij})]^{\text{tr}} A^{-1} + p([x_{ij}])I, \quad [x_{ij}] \in M_n.$$

In the case $n = 2$ there exists a ring automorphism $f : \mathbb{C} \rightarrow \mathbb{C}$, an additive function $p : M_2 \rightarrow \mathbb{C}$ and a linear mapping $L : M_2 \rightarrow M_2$ which leaves the subspace $\{\lambda I, \lambda \in \mathbb{C}\}$ invariant, such that ϕ is of the form

$$\phi([x_{ij}]) = L([f(x_{ij})]) + p([x_{ij}])I, \quad [x_{ij}] \in M_2.$$

Remarks. 1. This note contains also the proof for $n = 2$, the case that is exceptional, and was not considered in the previously mentioned papers.

2. If we add the assumption of injectivity, the result for $n \geq 3$ follows from [4].

3. Not only do we not need injectivity, in this particular case, studying the mappings on M_n , the proof is much shorter, and involves only simple linear algebra tools.

PROOF. We will show that ϕ is not “very far” from being linear. As ϕ preserves commutativity we have that

$$(4) \quad \phi(\alpha X) \leftrightarrow \phi(X)$$

for all complex numbers α , and $X \in M_n$. Let $\mu \in \mathbb{C}$, $\mu \neq 0$, be fixed. Since ϕ is surjective, we can get for every pair of indices i, j a matrix Z_{ij} with $\phi(Z_{ij}) = \mu E_{ij}$. All block matrices in the proof will be partitioned according to $\mathbb{C}^n = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \text{Span}(e_3, \dots, e_n)$ where $\{e_i, 1 \leq i \leq n\}$ is the standard basis of $\mathbb{C}^n$. The blocks that are not of dimension 1×1 will be denoted using capital letters. We have divided the proof into three steps.

Step 1. If $\phi(X) = \mu E_{ij} + \delta I$, $\delta \in \mathbb{C}$, $1 \leq i, j \leq n$, then $\phi(\alpha X)$, $\alpha \in \mathbb{C}$, is a sum of a scalar multiple of E_{ij} and a diagonal matrix $D = \text{diag}(d_1, d_2, \dots, d_n)$ satisfying $d_i = d_j$.

If $n = 2$ this is a straightforward consequence of (3) and (4). Let $n \geq 3$ and $i = j$. As E_{ii} is similar to E_{11} (by a permutation matrix) we may assume $i = 1$ with no loss of generality. Because of (4) $\phi(\alpha X) \leftrightarrow E_{11}$, and is therefore of the form

$$\phi(\alpha X) = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}.$$

By the same argument, we have that

$$\phi(\alpha Z_{22}) = \begin{bmatrix} b_{11} & 0 & B_{13} \\ 0 & b_{22} & 0 \\ B_{31} & 0 & B_{33} \end{bmatrix},$$

and by the additivity of ϕ

$$\phi(\alpha(X + Z_{22})) = \begin{bmatrix} a_{11} + b_{11} & 0 & B_{13} \\ 0 & a_{22} + b_{22} & A_{23} \\ B_{31} & A_{32} & A_{33} + B_{33} \end{bmatrix}.$$

For the same reason, $\phi(\alpha(X + Z_{22}))$ commutes with $\phi(X) + \phi(Z_{22})$, and therefore also with $E_{11} + E_{22}$. This forces A_{23} and A_{32} to be zero. Replacing Z_{22} by Z_{kk} , $k \geq 3$, in the previous consideration, we get that A_{33} is a diagonal matrix. In particular, $\phi(\alpha Z_{ii})$, $1 \leq i \leq n$, is a diagonal matrix for every $\alpha \in \mathbb{C}$.

Assume now $i \neq j$. Without loss of generality, we can fix $(i, j) = (1, 2)$. Since $\phi(\alpha X)$ commutes with E_{12} , we may write its block matrix as

$$\phi(\alpha X) = \begin{bmatrix} c_{11} & c_{12} & C_{13} \\ 0 & c_{11} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix}.$$

The matrix $\phi(\alpha(X + Z_{11} + Z_{22}))$, which is of the form

$$\begin{aligned} \phi(\alpha(X + Z_{11} + Z_{22})) &= \phi(\alpha X) + \phi(\alpha Z_{11}) + \phi(\alpha Z_{22}) \\ &= \begin{bmatrix} c_{11} & c_{12} & C_{13} \\ 0 & c_{11} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix} \\ &= \begin{bmatrix} c_{11} + a_{11} + b_{11} & c_{12} & C_{13} \\ 0 & c_{11} + a_{22} + b_{22} & 0 \\ 0 & C_{32} & C_{33} + A_{33} + B_{33} \end{bmatrix} \end{aligned}$$

commutes with $\phi(X + Z_{11} + Z_{22}) = \phi(X) + \phi(Z_{11}) + \phi(Z_{22})$, and consequently with $E_{12} + E_{11} + E_{22}$. This implies

$$C_{13} = 0 \quad \text{and} \quad C_{32} = 0.$$

In order to get that C_{33} is diagonal, we choose k , $3 \leq k \leq n$, and compute

$$\begin{aligned} \phi(\alpha(X + Z_{kk})) &= \phi(\alpha X) + \phi(\alpha Z_{kk}) \\ &= \begin{bmatrix} c_{11} & c_{12} & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} + \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \\ &= \begin{bmatrix} c_{11} + d_{11} & c_{12} & 0 \\ 0 & c_{11} + d_{22} & 0 \\ 0 & 0 & C_{33} + D_{33} \end{bmatrix}. \end{aligned}$$

We know that the last matrix commutes with $\phi(X + Z_{kk}) = \mu(E_{12} + E_{kk}) + \delta I$, and therefore also with E_{kk} . Moreover, D_{33} is diagonal, which yields the desired conclusion.

Step 2. If $\phi(X) = \mu E_{ij} + \delta I$, $\delta \in \mathbb{C}$, $1 \leq i, j \leq n$, then there exists a ring automorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ (independent of i, j and δ) and a complex valued function v_{ij} , such that for every $\alpha \in \mathbb{C}$ holds

$$(5) \quad \phi(\alpha X) = f(\alpha) \mu E_{ij} + v_{ij}(\alpha, \delta) I,$$

and v_{ij} is additive in the first argument.

It suffices to get (5) for $i = 1$ if $i = j$, and for $(i, j) = (1, 2)$ in the case $i \neq j$. Suppose first $i = j = 1$. If $n \geq 3$ choose $r \geq 3$. Using the additivity of ϕ and applying Step 1 leads to

$$\begin{aligned} \phi(\alpha(X + Z_{2r})) &= \phi(\alpha X) + \phi(\alpha Z_{2r}) \\ &= \text{diag}(a_1, \dots, a_n) + (\text{diag}(d_1, \dots, d_n) + u_{2r} E_{2r}), \quad d_2 = d_r, \end{aligned}$$

and because of (4), the matrix $\phi(\alpha(X + Z_{2r}))$ commutes with $E_{11} + E_{2r}$. Therefore, $a_r = a_2$ for all $r \geq 3$. Now, there exist functions u_{11} and v_{11} , both additive in the first argument, such that

$$(6) \quad \phi(\alpha X) = u_{11}(\alpha, \delta) E_{11} + v_{11}(\alpha, \delta) I.$$

Next, we shall derive a similar equation to the above one for $(i, j) = (1, 2)$. Let $\phi(X) = \mu E_{12} + \delta I$. For any k , $3 \leq k \leq n$, $\phi(\alpha(X + Z_{1k}))$ commutes with $E_{12} + E_{1k}$. This gives

$$0 = [\phi(\alpha(X + Z_{1k})), E_{12} + E_{1k}]_{1k} = \phi(\alpha X)_{11} - \phi(\alpha X)_{kk},$$

and, applying the assertion of Step 1, we get the existence of the functions u_{12} and v_{12} , such that

$$\phi(\alpha X) = u_{12}(\alpha, \delta) E_{12} + v_{12}(\alpha, \delta) I.$$

Hence, if $\phi(X) = \mu E_{ij} + \delta I$, $1 \leq i, j \leq n$, there exist functions u_{ij} and v_{ij} such that

$$(7) \quad \phi(\alpha X) = u_{ij}(\alpha, \delta) E_{ij} + v_{ij}(\alpha, \delta) I.$$

Functions u_{ij} and v_{ij} are additive in the first argument and unique. In the case $n = 2$, and $i \neq j$, the relation (7) is a straightforward consequence of Step 1. If $n = 2$ and $i = j$, $\phi(\alpha X)$ is a diagonal matrix by Step 1, and as $\mu \neq 0$ was fixed, we get (7) with u_{ii} and v_{ii} , $i = 1, 2$, unique.

Since ϕ is surjective, there exist matrices X_{ij} with $\phi(X_{ij}) = E_{ij}$. Fix the set $\{X_{ij}; \phi(X_{ij}) = E_{ij}\}$. In the previous consideration $\mu \neq 0$ was fixed but arbitrary. In particular, the application of (7) at $\mu = 1$ and $\delta = 0$ guaranties the existence of uniquely defined additive functions f_{ij} and g_{ij} with

$$(8) \quad \phi(\alpha X_{ij}) = f_{ij}(\alpha) E_{ij} + g_{ij}(\alpha) I.$$

We will now show that the functions f_{ij} are independent of i and j . Let $i \neq j$. By (4), and the additivity of ϕ , we have

$$\phi(\alpha(X_{ii} + X_{ij})) \leftrightarrow E_{ii} + E_{ij},$$

and

$$\phi(\alpha(X_{ii} + X_{ij})) = f_{ii}(\alpha) E_{ii} + f_{ij}(\alpha) E_{ij} + (g_{ii}(\alpha) + g_{ij}(\alpha)) I.$$

Therefore,

$$0 = [f_{ii}(\alpha) E_{ii} + f_{ij}(\alpha) E_{ij}, E_{ii} + E_{ij}] = (f_{ii}(\alpha) - f_{ij}(\alpha)) [E_{ii}, E_{ij}],$$

which implies $f_{ii} = f_{ij}$. Replacing E_{ij} by E_{ji} in the above computation, we also obtain $f_{ii} = f_{ji}$ for all $1 \leq i, j \leq n$, $n \geq 2$. From now on f will be written instead of f_{ij} .

Our next goal is to show that

$$(9) \quad u_{ij}(\alpha, \delta) = f(\alpha)\mu$$

for all $1 \leq i, j \leq n$. Let $\phi(X) = \mu E_{ij} + \delta I$, and take $k, k \neq j$. As $\phi(\alpha(X + X_{ik})) \leftrightarrow \mu E_{ij} + E_{ik}$, we have that

$$\begin{aligned} 0 &= [u_{ij}(\alpha, \delta) E_{ij} + f(\alpha) E_{ik}, \mu E_{ij} + E_{ik}] \\ &= (u_{ij}(\alpha, \delta) - \mu f(\alpha)) [E_{ij}, E_{ik}]. \end{aligned}$$

Certainly, we can always choose $k, k \neq j$, such that $[E_{ij}, E_{ik}] \neq 0$, and the desired conclusion now follows.

What is left is to show that f is multiplicative and surjective. For all complex α and β we have

$$(10) \quad \phi(\alpha\beta X_{12}) = f(\alpha\beta)E_{12} + g_{12}(\alpha\beta)I$$

and

$$\phi(\beta X_{12}) = f(\beta)E_{12} + g_{12}(\beta)I.$$

If $f(\beta) = 0$ the relation $\phi(\alpha(\beta X_{12} + X_{lk})) \leftrightarrow \phi(\beta X_{12}) + E_{lk}$, $1 \leq l, k \leq n$, implies that $\phi(\alpha\beta X_{12})$ is a scalar matrix, and thus $f(\alpha\beta) = 0$. Take now $\mu = f(\beta) \neq 0$, $\delta = g_{12}(\beta)$ and $(i, j) = (1, 2)$. Combining equations (7) with X being replaced by βX_{12} , and (9) we obtain

$$(11) \quad \phi(\alpha(\beta X_{12})) = f(\alpha)f(\beta)E_{12} + v_{12}(\alpha, g_{12}(\beta))I.$$

Comparing the last equation to (10) gives the multiplicativity of f .

It is routine to show that the set $\{X_{ij}, \phi(X_{ij}) = E_{ij}\}$, that has already been fixed before, forms a basis of M_n , $n \geq 2$. For details we refer the reader to [11, p. 208]. From the linear independence of the set $\{X_{ij}\}$, the relation (8) and the surjectivity of ϕ , the surjectivity of f is now easily obtained.

Note that for every $X \in M_n$ there exist unique numbers α_{ij} , $1 \leq i, j \leq n$, such that $X = \sum_{i,j} \alpha_{ij} X_{ij}$.

Step 3. There exists a surjective linear mapping $L : M_n \rightarrow M_n$, $n \geq 2$, that preserves commutativity, and an additive function q on M_n such that

$$\phi([x_{ij}]) = L([f(x_{ij})]) + q([x_{ij}])I.$$

Let us first define an additive mapping $\phi_1 : M_n \rightarrow M_n$,

$$\phi_1(X) = \phi_1\left(\sum_{i,j} \alpha_{ij} X_{ij}\right) = [f(\alpha_{ij})],$$

which is surjective (since f is surjective), and preserves commutativity because of

$$\begin{aligned} \phi(X) &= \phi\left(\sum_{i,j} \alpha_{ij} X_{ij}\right) \\ (12) \quad &= \phi_1(X) + \left(\sum_{i,j} g_{ij}(\alpha_{ij})\right)I \\ &= \phi_1(X) + p(X)I. \end{aligned}$$

Clearly, p just involved is additive. Furthermore, we observe that ϕ_1 is f -quasilinear as

$$\begin{aligned} \phi_1(\alpha X) &= \phi_1\left(\sum_{i,j} \alpha \alpha_{ij} X_{ij}\right) \\ &= f(\alpha) [f(\alpha_{ij})] \\ &= f(\alpha) \phi_1(X) \end{aligned}$$

for every $\alpha \in \mathbb{C}$. Let ψ denote the mapping on M_n defined by

$$\psi([x_{ij}]) = [f(x_{ij})]$$

which is additive, bijective and preserves commutativity in both directions. Finally, we define $L = \phi_1 \circ \psi^{-1}$, and observe that it is homogeneous. Indeed,

$$L(\alpha X) = \phi_1(\psi^{-1}(\alpha X)) = \phi_1(f^{-1}(\alpha)\psi^{-1}(X)) = \alpha L(X).$$

Moreover, L is additive, surjective and preserves commutativity which establishes the assertion of Step 3.

Since L is linear, surjective and preserves commutativity, we then clearly have $L(I) = cI$, for some $c \neq 0$. If $n = 2$, the relation $\phi_1 = L \circ \psi$ substituted in (12) gives the desired conclusion. If $n \geq 3$ we end the proof of the theorem by substituting the well known form of a surjective linear commutativity-preserving mapping [3], [13] in (12). $\square$

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