

On generalized recurrent Riemannian manifolds

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Abstract. In this paper we study the two 1-forms A and B appearing in the definition of the generalized recurrency of a Riemannian manifold. It is shown that for a non-zero constant scalar curvature, A is closed iff B is closed. For a nonconstant scalar curvature the 1-forms A and B cannot be closed both, unless A is collinear with B . Also, we have found out some results on a generalized conformally recurrent Riemannian manifold.

1. Introduction

In a recent paper [1] DE and GUHA introduced and studied a type of non-flat Riemannian space whose curvature tensor $K(X, Y, Z)$ of type $(1, 3)$ satisfies the condition:

$$(1.1) \quad (D_U K)(X, Y, Z) = A(U)K(X, Y, Z) + B(U)[g(Y, Z)X - g(X, Z)Y]$$

where A and B are two 1-forms, B is non-zero and D denotes the operator of covariant differentiation with respect to the metric tensor g . Such a space has been called a *generalized recurrent space*. Here B is called its associated 1-form. These spaces are related to the pseudo symmetric Riemannian spaces of M. C. CHAKI [7] and are special cases of the weakly symmetric Riemannian spaces of L. TAMÁSSY and T. Q. BINH [8]. If the 1-form $B(U)$ becomes zero in (1.1), then the space reduces to a recurrent space according to RUSE [6] and WALKER [5].

Contracting (1.1) with respect to ‘ X ’, we get

$$(1.2) \quad (D_U \text{Ric})(Y, Z) = A(U) \text{Ric}(Y, Z) + (n - 1)B(U)g(Y, Z).$$

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In this case, the Riemannian manifold M is called a *generalized Ricci recurrent space*, where A and B are as stated earlier. If the 1-form $B(U)$ becomes zero in (1.2), then the space reduces to a Ricci-recurrent space.

In this paper we have considered a non-flat n -dimensional Riemannian manifold in which the *conformal curvature tensor* C satisfies the condition:

$$(1.3) \quad (D_U C)(X, Y, Z) = A(U)C(X, Y, Z) + B(U)[g(Y, Z)X - g(X, Z)Y]$$

where A and B are two 1-forms, B is non-zero and the conformal curvature tensor C is defined by (see [2])

$$(1.4) \quad C(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-2}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \\ + g(Y, Z)R(X) - g(X, Z)R(Y)] \\ + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y].$$

Here K is the curvature tensor of type $(1, 3)$, Ric is the Ricci tensor of type $(0, 2)$, r is the scalar curvature and R is the Ricci tensor to type $(1, 1)$, defined by

$$(1.5) \quad \text{Ric}(X, Y) = g(R(X), Y).$$

Such an n -dimensional Riemannian manifold shall be called a *generalized conformally recurrent* Riemannian manifold. If the 1-form B is zero, then the manifold reduces to a conformally recurrent manifold [3].

The *conharmonic curvature tensor* N and the *concircular curvature tensor* W are given by [4]

$$(1.6) \quad N(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-2}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \\ + g(Y, Z)R(X) - g(X, Z)R(Y)]$$

and

$$(1.7) \quad W(X, Y, Z) = K(X, Y, Z) - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

respectively.

If the conharmonic curvature tensor N and concircular curvature tensor W satisfy the conditions:

$$(1.8) \quad (D_U N)(X, Y, Z) = A(U)N(X, Y, Z) + B(U)[g(Y, Z)X - g(X, Z)Y]$$

and

$$(1.9) \quad (D_U W)(X, Y, Z) = A(U)W(X, Y, Z) + B(U)[g(Y, Z)X - g(X, Z)Y],$$

respectively, where A and B are two 1-forms, then the Riemannian manifold is known as a *generalized conharmonically recurrent* manifold and a *generalized concircularly recurrent* manifold, respectively.

In Section 2 we have discussed the nature of 1-forms A and B and in Section 3 we have studied about a generalized conformally recurrent manifold.

2. Nature of the 1-forms A and B on a generalized recurrent space

Taking covariant derivative of (1.5) with respect to ' U ', we have

$$(2.1) \quad g((D_U R)(X), Y) = (D_U \text{Ric})(X, Y).$$

Using (1.2) in (2.1), we have

$$g((D_U R)(X), Y) = A(U) \text{Ric}(X, Y) + (n - 1)B(U)g(X, Y)$$

which yields

$$(2.2) \quad (D_U R)(X) = A(U)R(X) + (n - 1)B(U)(X).$$

Contracting (2.2) with respect to ' X ', we get

$$(2.3) \quad Ur = A(U)r + n(n - 1)B(U)$$

where r is the scalar curvature.

First we consider the case when the scalar curvature r is a constant and is different from zero. Then from (2.3), we have

$$(2.4) \quad A(U)r + n(n - 1)B(U) = 0.$$

Taking covariant derivative of (2.4) with respect to 'V', we get

$$(2.5) \quad (D_V A)(U)r + n(n-1)(D_V B)(U) = 0.$$

Interchanging U and V in (2.5) and then subtracting them, we get

$$[(D_V A)(U) - (D_U A)(V)]r + n(n-1)[(D_V B)(U) - (D_U B)(V)] = 0.$$

Thus we have

Theorem 1. *In a generalized recurrent space of non-zero constant scalar curvature r , the 1-form A is closed if and only if the 1-form B is closed.*

Next we consider the case when the scalar curvature r is not constant. From (2.3) it follows that

$$(2.6) \quad VUr = (D_V A)(U)r + A(U)(Vr) + n(n-1)(D_V B)(U).$$

Interchanging U and V in (2.6) and then subtracting, we get

$$(2.7) \quad [(D_V A)(U) - (D_U A)(V)]r + A(U)(Vr) - A(V)(Ur) \\ + n(n-1)[(D_V B)(U) - (D_U B)(V)] = 0.$$

Using (2.3) in (2.7), we get

$$[(D_V A)(U) - (D_U A)(V)]r + n(n-1)[(D_V B)(U) - (D_U B)(V)] \\ + n(n-1)[A(V)B(U) - A(U)B(V)] = 0.$$

Thus we can state the following theorem:

Theorem 2. *In a generalized recurrent space of non-constant scalar curvature r , the 1-forms A and B cannot be both closed, unless the 1-form A is collinear with the 1-form B .*

3. Generalized conformally recurrent Riemannian manifolds

Let M be a generalized recurrent smooth Riemannian manifold of dimension n . Taking covariant derivative of (1.4) with respect to 'U',

we get

$$(3.1) \quad (D_U C)(X, Y, Z) = (D_U K)(X, Y, Z) - \frac{1}{n-2} [(D_U \text{Ric})(Y, Z)X \\ - (D_U \text{Ric})(X, Z)Y + g(Y, Z)(D_U R)(X) - g(X, Z)(D_U R)(Y)] \\ + \frac{Ur}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y].$$

Using (1.1), (1.2) and (2.3) in (3.1), we get

$$(3.2) \quad (D_U C)(X, Y, Z) = A(U) \left[K(X, Y, Z) - \frac{1}{n-2} \{ \text{Ric}(Y, Z)X \right. \\ \left. - \text{Ric}(X, Z)Y - g(Y, Z)R(X) - g(X, Z)R(Y) \} \right. \\ \left. + \frac{r}{(n-1)(n-2)} \{ g(Y, Z)X - g(X, Z)Y \} \right] \\ + B(U) \left[g(Y, Z)X - g(X, Z)Y - \frac{2(n-1)}{n-2} \{ g(Y, Z)X - g(X, Z)Y \} \right. \\ \left. + \frac{n(n-1)}{(n-1)(n-2)} \{ g(Y, Z)X - g(X, Z)Y \} \right].$$

Using (1.4) in (3.2), we have

$$(D_U C)(X, Y, Z) = A(U)C(X, Y, Z)$$

which shows the condition of a conformally recurrent Riemannian manifold. Thus we get the following theorem:

Theorem 3. *A generalized recurrent Riemannian manifold is conformally recurrent for the same recurrence parameter.*

From (1.3) and (1.4) it follows that

$$(3.3) \quad (D_U K)(X, Y, Z) - A(U)K(X, Y, Z) - B(U)[g(Y, Z)X \\ - g(X, Z)Y] = \frac{1}{n-2} [(D_U \text{Ric})(Y, Z)X - (D_U \text{Ric})(X, Z)Y]$$

$$\begin{aligned}
& +g(Y, Z)(D_U R)(X) - g(X, Z)(D_U R)(Y) \\
& -A(U)\{\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y + g(Y, Z)R(X) - g(X, Z)R(Y)\} \\
& + \frac{r}{(n-1)(n-2)}[A(U)\{g(Y, Z)X - g(X, Z)(Y)\} \\
& -U\{g(Y, Z)R(X) - g(X, Z)R(Y)\}].
\end{aligned}$$

Permutting equation (3.3) twice with respect to U, X, Y ; adding the three equations and using Bianchi's second identity, we have

$$\begin{aligned}
(3.4) \quad & A(U)K(X, Y, Z) + A(X)K(Y, U, Z) + A(Y)K(U, X, Z) \\
& +B(U)[g(Y, Z)X - g(X, Z)Y] + B(X)[g(U, Z)Y - g(Y, Z)U] \\
& +B(Y)[g(X, Z)U - g(U, Z)X] \\
& + \frac{1}{n-2}[(D_U \text{Ric})(Y, Z)X - (D_U \text{Ric})(X, Z)Y + g(Y, Z)(D_U R)(X) \\
& -g(X, Z)(D_U R)(Y) + (D_X \text{Ric})(U, Z)Y \\
& -(D_X \text{Ric})(Y, Z)U + g(U, Z)(D_X R)(Y) \\
& -g(Y, Z)(D_X R)(U) + (D_Y \text{Ric})(X, Z)U - (D_Y \text{Ric})(U, Z)X \\
& +g(X, Z)(D_Y R)(U) - g(U, Z)(D_Y R)(X) \\
& -A(U)\{\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y + g(Y, Z)R(X) - g(X, Z)R(Y)\} \\
& -A(X)\{\text{Ric}(U, Z)Y - \text{Ric}(Y, Z)U + g(U, Z)R(Y) - g(Y, Z)R(U)\} \\
& -A(Y)\{\text{Ric}(X, Z)U - \text{Ric}(U, Z)X + g(X, Z)R(U) - g(U, Z)R(X)\}] \\
& + \frac{r}{(n-1)(n-2)}[A(U)\{g(Y, Z)X - g(X, Z)(Y)\} \\
& -U\{g(Y, Z)X - g(X, Z)Y\}] \\
& +A(X)\{g(U, Z)Y - g(Y, Z)U\} - X\{g(U, Z)Y - g(Y, Z)U\} \\
& +A(Y)\{g(X, Z)U - g(U, Z)X\} - Y\{g(X, Z)U - g(U, Z)X\} = 0.
\end{aligned}$$

Contracting (3.4) with respect to 'X', we get

$$\begin{aligned}
(3.5) \quad & A(U) \text{Ric}(Y, Z) - A(Y) \text{Ric}(U, Z) + K(Y, U, Z, p) \\
& +(n-1)B(U)g(Y, Z) + B(Y)g(U, Z) - B(U)g(Y, Z) \\
& +(1-n)B(Y)g(U, Z) + \frac{1}{n-2}[(n-1)(D_U \text{Ric})(Y, Z)
\end{aligned}$$

$$\begin{aligned}
& +g(Y, Z)(Ur) - g(D_U R)(Y, Z) + (D_Y \text{Ric})(U, Z) \\
& - (D_U \text{Ric})(Y, Z) + \frac{1}{2}g(U, Z)(Yr) - \frac{1}{2}g(Y, Z)(Ur) \\
& + (1-n)(D_Y \text{Ric})(U, Z) + g(D_Y R)(U, Z) - g(U, Z)(Yr) \\
& - (n-1)A(U) \text{Ric}(Y, Z) - A(U)g(Y, Z)r + A(U) \text{Ric}(Y, Z) \\
& - A(Y) \text{Ric}(U, Z) + A(U) \text{Ric}(Y, Z) - A(R(Y))g(U, Z) \\
& + A(R(U))g(Y, Z) + (n-1)A(Y) \text{Ric}(U, Z) - A(Y) \text{Ric}(U, Z) \\
& + A(Y)g(U, Z)r + \frac{r}{(n-1)(n-2)}[(n-1)A(U)g(Y, Z) \\
& - (n-1)g(Y, Z)U + A(Y)g(U, Z) - A(U)g(Y, Z) - ng(U, Z)Y \\
& + ng(Y, Z)U + (1-n)A(Y)g(U, Z) + (n-1)g(U, Z)Y] = 0
\end{aligned}$$

where p is a vector field defined by

$$(3.6) \quad g(X, p) = A(X).$$

Using (1.5) in (3.5), we have

$$\begin{aligned}
& A(U)R(Y) - A(Y)R(U) - K(Y, U, p) + (n-2)[B(U)Y - B(Y)U] \\
& + \frac{1}{n-2}[(n-1)(D_U R)(Y) + Y(Ur) - (D_U R)(Y) + (D_Y R)(U) \\
& - (D_U R)(Y) + \frac{1}{2}U(Yr) - \frac{1}{2}Y(Ur) + (1-n)(D_Y R)(U) + (D_Y R)(U) \\
& - U(Yr) - (n-1)A(U)R(Y) - A(U)Yr + A(U)R(Y) - A(Y)R(U) \\
& + A(U)R(Y) - A(R(Y))U + A(R(U))Y + (n-1)A(Y)R(U) \\
& - A(Y)R(U) + A(Y)Ur] + \frac{r}{(n-1)(n-2)}[(n-1)A(U)Y - (n-1)YU \\
& + A(Y)U - A(U)Y - nUY + nYU + (1-n)A(Y)U + (n-1)UY] = 0
\end{aligned}$$

or

$$\begin{aligned}
(3.7) \quad & K(Y, U, p) - (n-2)[B(U)Y - B(Y)U] \\
& = \frac{1}{n-2}[(n-3)(D_U R)(Y) - (n-3)(D_Y R)(U) \\
& + A(U)R(Y) - A(Y)R(U) + A(R(U))Y - A(R(Y))U] \\
& + \frac{r}{(n-1)(n-2)}[A(Y)U - A(U)Y].
\end{aligned}$$

Contracting (3.7) with respect to 'Y', we get

$$\begin{aligned} \text{Ric}(U, p) - (n-1)(n-2)B(U) &= \frac{1}{n-2} \left[(n-3)Ur - \frac{1}{2}(n-3)(Ur) \right. \\ &\quad \left. + A(U)r - A(R(U)) + (n-1)A(R(U)) \right] + \frac{r}{(n-1)(n-2)} [(n-1)A(U)] \end{aligned}$$

or

$$2A(U)r + 2(n-1)(n-2)^2B(U) = (n-3)(Ur).$$

Thus we have

Theorem 4. *The necessary and sufficient condition that the scalar curvature r of a generalized conformally recurrent Riemannian manifold be constant is that*

$$A(U)r + (n-1)(n-2)^2B(U) = 0.$$

From (1.4), (1.6) and (1.7), we have

$$(3.8) \quad C(X, Y, Z) = N(X, Y, Z) + \frac{n}{n-2} [K(X, Y, Z) - W(X, Y, Z)].$$

Taking covariant derivative of (3.8) with respect to 'U', we get

$$(3.9) \quad \begin{aligned} (D_U C)(X, Y, Z) &= (D_U N)(X, Y, Z) \\ &+ \frac{n}{n-2} [(D_U C)(X, Y, Z) - (D_U W)(X, Y, Z)]. \end{aligned}$$

Using (1.1) in (3.9), we get

$$(3.10) \quad \begin{aligned} (D_U C)(X, Y, Z) - (D_U N)(X, Y, Z) &+ \frac{n}{n-2} (D_U W)(X, Y, Z) \\ &= \frac{n}{n-2} [A(U)K(X, Y, Z) + B(U)\{g(Y, Z)X - g(X, Z)Y\}]. \end{aligned}$$

From (3.10) it is evident that if any two of the equations (1.3), (1.8) and (1.9) hold then the third also holds.

This leads the following:

Theorem 5. *Let M be a generalized recurrent Riemannian manifold of dimension n . If any two of the following hold then the third also holds:*

- (i) *It is generalized conformally recurrent manifold.*
- (ii) *It is generalized conharmonically recurrent manifold.*
- (iii) *It is generalized concircularly recurrent manifold.*

For the same recurrence parameters.

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