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Submanifolds of a quasi constant curvature manifold

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Abstract. In this paper, we study submanifolds in a quasi constant curvature space and generalize a result due to Chern–do Carmo–Kobayashi and Li–Li.

0. Introduction

Let M be an n-dimensional compact minimal submanifold immersed into the $(n+p)$-dimensional unit sphere S^{n+p}. Denote by S the square of the length of the second fundamental form of M. The following result is well known ([1, 2]):

Theorem A. *Let M be an n-dimensional compact minimal submanifold in S^{n+p}. If S satisfies*

$$S \le \frac{n}{1+\frac{1}{2}\operatorname{sgn}(p-1)},$$

then M is a totally geodesic submanifold, and the Clifford torus or the Veronese surface is in S^4, where $\operatorname{sgn}(x)=1$ *for* $x>0$ *and* $\operatorname{sgn}(x)=0$ *for* $x\le 0$.

The following definition was introduced by Chen and Yano [3].

Definition. A Riemannian manifold is said to be a quasi constant curvature manifold if its curvature tensor satisfies

$$(*)\qquad K_{ABCD} = a\,(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}) + b(\delta_{AC}v_Bv_D-\delta_{AD}v_Bv_D+\delta_{BD}v_Av_C-\delta_{BC}v_Av_D),$$

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where a, b are some scalar functions and v_A the component of a unit vector field which is called the generator of the manifold.

By the definition we see that when $b \equiv 0$, the quasi constant curvature manifold becomes a constant curvature manifold.

From now on, we use of the following convention on the ranges of the indices:

$$1 \leq A, B, \ldots \leq n+p;\ 1 \leq i, j, \ldots \leq n;\ n+1 \leq \alpha, \beta, \ldots \leq n+p.$$

The purpose of this paper is to study the case that the ambient space is a quasi constant curvature manifold N^{n+p} and to generalize Theorem A. We obtain

Theorem 1. *Let M be an n-dimensional compact minimal submanifold in an $(n+p)$-dimensional quasi constant curvature manifold N^{n+p} and let a, b be constants. Suppose that*
(a) *$a > 0$, the generator is orthogonal to M and*

$$S < \frac{na}{1 + \frac{1}{2}\operatorname{sgn}(p-1)},$$

or else that
(b) *$na + b - n|b| > 0$, the generator is parallel to M, and*

$$S < \frac{na + b - n|b|}{1 + \frac{1}{2}\operatorname{sgn}(p-1)}.$$

Then each of these two sets of conditions implies that M is a totally geodesic submanifold.

Theorem 2. *Let M be an n-dimensional compact minimal submanifold in an $(n+p)$-dimensional quasi constant curvature manifold N^{n+p}. Suppose a is a positive number. If the generator is orthogonal to M and*

$$S = \frac{na}{1 + \frac{1}{2}\operatorname{sgn}(p-1)},$$

then either $M = S^m\left(\sqrt{\frac{m}{(n-m)a+m}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{ma+n-m}}\right)$ or $n = p = 2$ and with respect to an adapted dual orthonormal frame ω_1, ω_2, ω_3, ω_4, *the connection form* (ω_{AB}) *of* N^4*, restricted to* M*, is given by*

$$\begin{pmatrix} 0 & \omega_{12} & g\omega_1 & g\omega_2 \\ \omega_{21} & 0 & -g\omega_2 & g\omega_1 \\ -g\omega_1 & g\omega_2 & 0 & 2\omega_{12} \\ -g\omega_2 & -g\omega_1 & -2\omega_{12} & 0 \end{pmatrix}, \quad g = \sqrt{\frac{1}{3}a}\,.$$

Remark. When $a = 1$ and $b = 0$, from Theorems 1, 2 we can get Theorem A immediately.

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1. Fundamental formulas

Let M be an n-dimensional compact minimal submanifold immersed in the $(n+p)$-dimensional quasi constant curvature manifold N^{n+p} with the curvature tensor as $(*)$. Suppose that $\{e_A\}$ is an orthonomal frame field on N^{n+p} such that $e_1, \dots, e_n$ are tangent to M, and let $\{\omega_A\}$ be the dual frame field. Then the structure equations of N^{n+p} are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_B,$$

where K_{ABCD} satisfies $(*)$.

Restricting these forms to M, we have

$$\omega_\alpha = 0, \quad \omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji},$$

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{1.1}$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2} \sum_{kl} R_{\alpha\beta kl} \omega_k \wedge \omega_l,$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h^\alpha_{ki} h^\beta_{il} - h^\beta_{ki} h^\alpha_{il}).$$

We call $h = \sum_{ij\alpha} h^\alpha_{ij} \omega_i \omega_j e_\alpha$ and $\xi = \frac{1}{n} \sum_{i\alpha} h^\alpha_{ii} e_\alpha$ the second fundamental form and the mean curvature vector of the immersion, respectively. M is said to be minimal, if $\xi \equiv 0$. Denote by $S = \sum_{ij\alpha} (h^\alpha_{ij})^2$ the square of the length of h. Define h^α_{ijk} and h^α_{ijkl} by

$$\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{kj} \omega_{ki} - \sum_k h^\alpha_{ik} \omega_{kj} - \sum_\beta h^\beta_{ij} \omega_{\beta\alpha}$$

and by

$$\sum_l h^\alpha_{ijkl} \omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ljk} \omega_{li} - \sum_l h^\alpha_{ilk} \omega_{lj} - \sum_l h^\alpha_{ijl} \omega_{lk} - \sum_\beta h^\beta_{ijk} \omega_{\beta\alpha},$$

respectively. We know that

$$h^\alpha_{ijk} - h^\alpha_{ikj} = K_{\alpha ikj} = -K_{\alpha ijk} \tag{1.2}$$

and

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl} + \sum_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{1.3}$$

2. Proofs of the theorems

In order to prove our theorems, we need the following

Lemma 1 ([1, 2]). *Let H_α, $\alpha \geq 2$ be symmetric $(n \times n)$-matrices, $S_\alpha = \operatorname{tr} H^2_\alpha$, $S = \sum_\alpha S_\alpha$. Then*

$$\sum_{\alpha,\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \geq -\left[1 + \tfrac{1}{2} \operatorname{sgn}(p-1)\right] S^2,$$

and equality holds if and only if all $H_\alpha = 0$ or there exist two H_α's different from zero. Moreover, if $H_{n+1} \neq 0$, $H_{n+2} \neq 0$, $H_\alpha = 0$, $\alpha \neq n+1, n+2$, then $S_{n+1} = S_{n+2}$ and there exists an orthogonal $(n \times n)$ matrix T such that

$$TH_{n+1}{}^tT = \begin{pmatrix} f & 0 & 0 \\ 0 & -f & \\ 0 & & 0 \end{pmatrix}, \quad TH_{n+2}{}^tT = \begin{pmatrix} 0 & f & 0 \\ f & 0 & \\ 0 & & 0 \end{pmatrix},$$

where $f = \sqrt{\frac{S_1}{2}}$.

If a and b are constants and M is minimal, then using $(*)$, (1.1)–(1.3) we get

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + \sum_{i,j,\alpha} h^\alpha_{ij} \Delta h^\alpha_{ij} = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 \\ &\quad + \sum_{i,j,k,m,\alpha} h^\alpha_{ij} h^\alpha_{mk} R_{mijk} + \sum_{i,j,k,m,\alpha} h^\alpha_{ij} h^\alpha_{im} R_{mkjk} \\ &\quad + \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij} h^\beta_{ki} R_{\beta\alpha jk} + \sum_{i,j,k,\alpha} h^\alpha_{ij} \nabla_k K_{\alpha ikj} \\ &\quad + \sum_{i,j,k,\alpha} h^\alpha_{ij} \nabla_j K_{\alpha kki} = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + naS + bS \sum_k v_k^2 \\ &\quad + nb \sum_{i,j,m,\alpha} h^\alpha_{ij} h^\alpha_{im} v_m v_j - n \sum_{i,j,\alpha} h^\alpha_{ij} \nabla_j (b v_\alpha v_j) \\ &\quad + \sum_{\alpha,\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta} (\operatorname{tr} H_\alpha H_\beta)^2. \end{aligned} \tag{2.1}$$

Now, if we assume that the generator $v = \sum_A v_A e_A$ is orthogonal to M, then we see that $v_i = 0$ and (2.1) becomes

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + naS + \sum_{\alpha,\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ &\quad - \sum_{\alpha,\beta} (\operatorname{tr} H_\alpha H_\beta)^2. \end{aligned} \tag{2.2}$$

Applying Lemma 1 to (2.2) we get

$$\frac{1}{2}\Delta S \geq \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + naS - \left[(1 + \tfrac{1}{2}\,\mathrm{sgn}(p-1)\right] S^2 \tag{2.3}$$
$$\geq naS - \left[1 + \tfrac{1}{2}\,\mathrm{sgn}(p-1)\right] S^2.$$

When a is a positive number, it follows from the compactness of M and (2.3) that if

$$S \leq \frac{na}{1 + \frac{1}{2}\,\mathrm{sgn}(p-1)}, \tag{2.4}$$

then S is a constant and (2.4) leads to

$$\left\{na - \left[1 + \tfrac{1}{2}\,\mathrm{sgn}(p-1)\right] S\right\} S = 0. \tag{2.5}$$

Thus, if

$$S < \frac{na}{1 + t\frac{1}{2}\,\mathrm{sgn}(p-1)},$$

then from (2.5) we see that $S = 0$, hence M is totally geodesic.

If the generator $v = \sum_A v_A e_A$ is parallel to M, then we see that $v_\alpha = 0$ and $\sum_i v_i^2 = 1$. Applying Lemma 1 to (2.1) we get

$$\begin{aligned}\frac{1}{2}\Delta S \geq \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + naS + bS + nb \sum_{i,j,m,\alpha} h^\alpha_{ij} h^\alpha_{im} v_m v_j \\ - \left[1 + \tfrac{1}{2}\,\mathrm{sgn}(p-1)\right] S^2.\end{aligned} \tag{2.6}$$

We claim that, for any α

$$\sum_{i,j,m} h^\alpha_{ij} h^\alpha_{im} v_m v_j \leq \sum_{i,j} (h^\alpha_{ij})^2.$$

In fact, since both sides of the formula above are independent of e_i, we can choose $e_1, \ldots, e_n$ such that $h^\alpha_{ij} = h^\alpha_{ii}\delta_{ij}$ for fixed α, and hence

$$\sum_{ijm} h^\alpha_{ij} h^\alpha_{im} v_m v_j = \sum_i (h^\alpha_{ii})^2 v_i^2 \leq \sum_i (h^\alpha_{ii})^2 \sum_i v_i^2 = \sum_{ij} (h^\alpha_{ij})^2.$$

Thus

$$\sum_{i,j,m,\alpha} h^\alpha_{ij} h^\alpha_{im} v_m v_j \le \sum_{i,j,\alpha} (h^\alpha_{ij})^2 = S. \tag{2.7}$$

Combining (2.7) with (2.6) we get

$$\begin{aligned} \frac{1}{2}\Delta S &\ge \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + naS + nS - nbS - \left[1 + \tfrac{1}{2}\operatorname{sgn}(p-1)\right] S^2 \\ &\ge \left\{ na + b - n|b| - \left[1 + \tfrac{1}{2}\operatorname{sgn}(p-1)\right] S \right\} S. \end{aligned} \tag{2.8}$$

Using the same arguments as above we see that if

$$S < \frac{na + b - n|b|}{1 + \frac{1}{2}\operatorname{sgn}(p-1)},$$

then M is totally geodesic. This completes the proof of Theorem 1.

By (1.1), the scalar curvature of M satisfies

$$R = (n-1)\left(na + 2b \sum_i v_i^2 \right) - S,$$

which shows that if the generator is orthogonal to M, then

$$R = (n-1)na - S \tag{2.9}$$

and if the generator is parallel to M, then we have

$$R = (n-1)(na + 2b) - S. \tag{2.10}$$

Therefore, combining (2.9) with (2.10), Theorem 1 implies the

Corollary. *Let M be an n-dimensional compact minimal submanifold in an $(n+p)$-dimensional quasi constant curvature manifold N^{n+p} and let a and b be constants. Suppose that*

(a) *$a > 0$, the generator is orthogonal to M, and*

$$R > na \left[n - 1 - \frac{1}{1 + \frac{1}{2}\operatorname{sgn}(p-1)} \right]$$

or else that

(b) $na+b-n|b|>0$, *the generator is parallel to* M *and*

$$R>(n-1)(na+2b)-\frac{na+b-n|b|}{1+\frac{1}{2}\operatorname{sgn}(p-1)}.$$

Then each of these two sets of conditions implies that M *is a totally geodesic submanifold.*

Proof of Theorem 2. If

$$S=\frac{na}{1+\frac{1}{2}\operatorname{sgn}(p-1)}, \tag{2.11}$$

and $p=1$, (2.3) and (2.11) show that $\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2=0$. We can choose $e_1,\dots,e_n$ so that $h^{n+1}_{ij}=h^{n+1}_{ii}\delta_{ij}$. Since $v_i=0$ and using the same method as in [1] we get

$$h^{n+1}_{ii}h^{n+1}_{jj}=-a \tag{2.12}$$

for $h^{n+1}_{ii}\neq h^{n+1}_{jj}$. (2.12) implies that M has two distinct principal curvatures λ, μ and these satisfy

$$m\lambda+(n-m)\mu=0,\quad m\lambda^2+(n-m)\mu^2=na,\quad \lambda\mu=-a,$$

hence

$$\lambda^2+1=\frac{(n-m)a+m}{m},\quad \mu^2+1=\frac{ma+n-m}{n-m}.$$

So by the same arguments as in [1] we see that M is locally congruent to $S^m\left(\sqrt{\frac{m}{(n-m)a+m}}\right)\times S^{n-m}\left(\sqrt{\frac{n-m}{ma+n-m}}\right)$.

On the other hand, for $p\geq 2$, (2.2), (2.3) and (2.6) imply that $\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2=0$ and the equality

$$\sum_{\alpha,\beta}\operatorname{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2-\sum_{\alpha,\beta}(\operatorname{tr}H_\alpha H_\beta)^2=-\frac{3}{2}S^2$$

holds. Thus applying Lemma 1 we may assume that

$$H_{n+1}=\begin{pmatrix} g & 0 & 0\\ 0 & -g & \\ 0 & & 0\end{pmatrix},\quad H_{n+2}=\begin{pmatrix} 0 & g & 0\\ g & 0 & \\ 0 & & 0\end{pmatrix},$$

where we have $g = \sqrt{\frac{S_1}{2}} = \sqrt{\frac{1}{6}na}$. Noting that $v_i = 0$, and using the method of [1], we get $n = p = 2$. Thus we have

$$\omega_{13} = \omega_{24} = \sqrt{\frac{1}{3}}a\omega_1, \quad \omega_{14} = \sqrt{\frac{1}{3}}a\omega_2, \quad \omega_{23} = -\sqrt{\frac{1}{3}}a\omega_2, \quad \omega_{34} = 2\omega_{12},$$

and so with respect to an adapted dual orthonormal frame field ω_1, ω_2, ω_3, ω_4, the connection forms $\{\omega_{AB}\}$ of N^4, restricted to M, are given by

$$\begin{pmatrix} 0 & \omega_{12} & g\omega_1 & g\omega_2 \\ \omega_{12} & 0 & -g\omega_2 & g\omega_1 \\ -g\omega_1 & g\omega_2 & 0 & 2\omega_{12} \\ -g\omega_2 & -g\omega_1 & -2\omega_{12} & 0 \end{pmatrix}, \quad g = \sqrt{\frac{1}{3}a}.$$

This completes the proof of Theorem 2. $\square$

When $a = 1$, $b = 0$, it follows from Theorems 1 and 2 that we can immediately prove Theorem A of the Introduction.

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