

Characterization of group homomorphisms having values in an inner product space

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Abstract. In this note we prove that for functions $f : G \rightarrow E$ from a group G to an inner product space E , the inequality $\|f(xy)\| \geq \|f(x) + f(y)\|$ ($x, y \in G$) implies $f(xy) = f(x) + f(y)$ ($x, y \in G$), and ask the open question: Is this statement true also for strictly convex normed spaces E ?

1. The main result

In this note we prove the following

Theorem. *For functions $f : G \rightarrow E$ from a group G to a real or complex inner product space E , the inequality*

$$(1) \quad \|f(xy)\| \geq \|f(x) + f(y)\| \quad (x, y \in G)$$

implies

$$(2) \quad f(xy) = f(x) + f(y) \quad (x, y \in G).$$

PROOF. Let n denote the identity of the group G . With $x = y = n$ in (1) we get $f(n) = 0$, and with $y = x^{-1}$ we then have $f(x^{-1}) = -f(x)$ for all $x \in G$. Inequality (1) can be rewritten as

$$(3) \quad \|f(x)\|^2 + 2 \operatorname{Re}\langle f(x), f(y) \rangle + \|f(y)\|^2 \leq \|f(xy)\|^2.$$

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Replacing x and y by xy and y^{-1} , respectively, we get

$$\begin{aligned} \|f(xy)\|^2 + 2\operatorname{Re}\langle f(xy), f(y^{-1}) \rangle + \|f(y^{-1})\|^2 &\leq \|f(x)\|^2, \quad \text{i.e.,} \\ \|f(xy)\|^2 - 2\operatorname{Re}\langle f(xy), f(y) \rangle + \|f(y)\|^2 &\leq \|f(x)\|^2. \end{aligned}$$

Adding this inequality to (3) and dividing by 2 we have

$$(4) \quad \operatorname{Re}\langle f(x) + f(y) - f(xy), f(y) \rangle \leq 0.$$

Replacing in (3) x and y by x^{-1} and xy , respectively, we obtain

$$\begin{aligned} \|f(x^{-1})\|^2 + 2\operatorname{Re}\langle f(x^{-1}), f(xy) \rangle + \|f(xy)\|^2 &\leq \|f(y)\|^2, \quad \text{i.e.,} \\ \|f(x)\|^2 - 2\operatorname{Re}\langle f(x), f(xy) \rangle + \|f(xy)\|^2 &\leq \|f(y)\|^2. \end{aligned}$$

Combining also this inequality with (3) yields

$$(5) \quad \operatorname{Re}\langle f(x) + f(y) - f(xy), f(x) \rangle \leq 0.$$

Replacing here x and y by xy and y^{-1} , respectively, we get

$$(6) \quad \operatorname{Re}\langle f(xy) - f(y) - f(x), f(xy) \rangle \leq 0.$$

Finally, adding (4), (5), and (6), we obtain

$$\|f(x) + f(y) - f(xy)\|^2 \leq 0,$$

which implies (2). □

2. Comments

1. The foregoing proof is essentially as in [5], where abelian groups G had been treated. In that case, of course, (5) is the same as (4), when interchanging x and y . In the non-abelian case it is clear that (5) can be derived in the same manner as (4) by considering in G the group operation $(x, y) \rightarrow yx$.

2. In the talk mentioned in the title of [5] (this talk is called “paper” in the title of [4]) only the particular case $G = E = \mathbb{R}$ of the theorem had been treated (and the proof was more complicated).

3. KUREPA [4] proved the theorem for functions $f : G \rightarrow E$ satisfying the supplementary condition

$$f(xyz) = f(xzy) \quad (x, y, z \in G).$$

From [4] it is also clear that the theorem does not hold for arbitrary semi-groups G : Consider the interval $[0, +\infty[$ with operation $+$. The function $f : [0, +\infty[\rightarrow \mathbb{R}$ given by

$$(7) \quad f(x) = x^2 \quad (x \geq 0)$$

satisfies $|f(x+y)| \geq |f(x)+f(y)|$, but we do not have $f(x+y) = f(x)+f(y)$ for all $x \geq 0, y \geq 0$.

4. Open problems. Let us start with an open problem from the literature. In [2] FISCHER and MUSZÉLY conjectured the following.

Let E be a strictly convex normed space, G an arbitrary semi-group, and $f : G \rightarrow E$ such that

$$\|f(xy)\| = \|f(x) + f(y)\| \quad (x, y \in G).$$

Then (2) holds.

The conjecture is true for inner product spaces E (FISCHER and MUSZÉLY [2]), and it is true for groups G (GER [3]). Both papers also contain other interesting results; already from [2] it follows that the conjecture is false for normed spaces E , which are not strictly convex (cf. [1] for the special case $E = C[0, 1]$). These facts, together with counterexample (7) for the additive semi-group $G = [0, +\infty[$ lead to the open question formulated in the abstract.

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