Publ. Math. Debrecen
42 / 1–2 (1993), 1–10

Dynamical connections and higher-order Lagrangian systems

By D. OPRIS and F.C. KLEPP* (Timişoara)

Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. We show that if ξ is a $(2r)$-order differential equation (semispray) on the $(2r-1)$-jet bundle $\mathcal{J}^{2r-1}Q$ whose paths are solutions of the non-autonomous Lagrange equations, then there is a connection Γ on $\mathcal{J}^{2r-1}Q$ whose paths are also solutions of the same equations. Moreover, Γ is a connection whose associated semispray is precisely ξ. This is an extension to higher-order Lagrangian dynamics of a previous result given by M. de Leon and P. Rodrigues [3].

1. Preliminaries

Throughout the text we shall keep in mind the results, definitions and notations previously introduced in [1], [2]. All structures and functions are assumed to be smooth. Let M be an m-dimensional manifold, called configuration manifold and Γ an $(r+1)$-order differential equation field on M. We recall here that Γ generates on T^rM two projectors: $A : T(T^rM) \to \text{Hor }(T^rM)$ and $B : T(T^rM) \to \text{Ver }(T^rM)$ such that $T(T^rM) = \text{Hor }(T^rM) \oplus \text{Ver}(T^rM)$ [1]. If $\bar{\xi}$ is an arbitrary semispray, i.e. an $(r+1)$-order differential equation field, then $\xi = A(\bar{\xi})$ is a semispray on T^rM which does not depend on the choice of $\bar{\xi}$. We call ξ the associated semispray of Γ. In the non-autonomous situation the relation between connections and semisprays becomes much more simple, as we will show below.

Let $\pi : Q \to X$ be a fibered manifold. In the following we assume that X is a connected real 1-dimensional manifold (i.e. $X = R$ or $X = S^1$) and Q is a real $(n+1)$-dimensional manifold. The r-order jet-prolongation is denoted by $\pi^r : \mathcal{J}^rQ \to X$. We denote by VQ the vertical bundle

*This paper was presented at the Conference on Finsler Geometry and its Applications to Physics and Control Theory, August 26–31, 1991, Debrecen, Hungary.

of Q, i.e. the vector subbundle of TQ defined as $VQ = \mathrm{Ker}(\pi')$ and by $V^0\mathcal{J}^rQ$ the vertical bundle of $\mathcal{J}^rQ$, defined as $V^0\mathcal{J}^rQ = \mathrm{Ker}(\pi_0^{r\prime})$, where $\pi_0^r : \mathcal{J}^rQ \to Q$ is the canonical projection.

Let $\pi_r^{r+1} : \mathcal{J}^{r+1}Q \to \mathcal{J}^rQ$ be the canonical projection, and $\eta_r : V\mathcal{J}^rQ \to \mathcal{J}^rQ$, the usual projection of the vertical bundle of $\mathcal{J}^rQ$. Then there exists $T : \mathcal{J}^{r+1}Q \to V\mathcal{J}^rQ$ satisfying $\eta_r \cdot \pi = \pi_r^{r+1}$ [1]. We use the map T to construct a differential operator d_T, which maps each function on $\mathcal{J}^rQ$ to a function on $\mathcal{J}^{r+1}Q$, and is called the partial time derivative. It follows that d_T is represented in coordinates $(t, q^i_{(k)})$, $1 \leq i \leq n$, $0 \leq k \leq r+1$ by the operator:

$$d_T = \sum_{h=0}^{r} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}}.$$

The operator d_T is a derivation in the sense that

$$d_T(f_1 \cdot f_2) = (d_T f_1)(\pi_r^{r+1*} f_2) + (\pi_r^{r+1*} f_1) d_T f_2; \quad f_1, f_2 \in \mathcal{F}(\mathcal{J}^rQ).$$

The extended operator, which we also denote d_T, bears the same relation to the operator on functions as a Lie derivative operator does to the action of a vector field on functions. Thus so far as its action on 1-forms is concerned (and this will be sufficient for our purposes), d_T satisfies the following rules:

1) for any 1-form α on $\mathcal{J}^rQ$, $d_T\alpha$ is an 1-form on $\mathcal{J}^{r+1}Q$
2) $d_T \cdot d = d \cdot d_T$
3) for any function f on $\mathcal{J}^rQ$,

$$d_T(f\alpha) = (d_T f)(\pi_r^{r+1*}\alpha) + (\pi_r^{r+1*} f) d_T(\alpha).$$

In particular, the coordinate 1-forms satisfy:

$$d_T(dq^i_{(k)}) = dq^i_{(k+1)}; \quad 0 \leq k \leq r.$$

We shall now define the lifts of a function f in Q to $\mathcal{J}^rQ$. For $k = 0, 1, \ldots, r$ we define the $k+1$ lift f_{k+1} of a function f to $\mathcal{J}^rQ$, by $f_{k+1} = d(f_k)$, $k = 0, 1, \ldots, r$ where $f_0 = f$ is a function on Q.

Let $\xi \in VQ_0$ be a vector field. The vector field $\xi^v \in V_0\mathcal{J}^rQ$ given by

$$\xi^v(f_r) = \pi_{r-1}^{r*}\xi(f_{r-1})$$

for any $f \in \mathcal{F}(Q)$, is called the vertical lift of ξ.

Using the vertical lift and the map $T : \mathcal{J}^rQ \to V\mathcal{J}^{r-1}Q$ we construct a canonical vector field on $\mathcal{J}^rQ$ as follows: $C = T^v$.

The coordinate representation of C is:

$$C = \sum_{h=0}^{r-1} (h+1) q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h+1)}}$$

where the vector field C is a generalization of the Liouville field or the dilation field on TQ.

We may also use the vertical lift construction to define an (1,1)-type tensor field S on $\mathcal{J}^r Q$, given by

$$S(\xi) = \left[\left(\pi^r_{r-1}\right)' \xi\right]^v, \qquad \forall \xi \in V\mathcal{J}^r Q.$$

The coordinate representation of S is

$$S = \sum_{h=0}^{r-1} (h+1) \frac{\partial}{\partial q^i_{(h+1)}} \otimes dq^i_{(h)}.$$

Therefore we transport the geometric structures defined on $V\mathcal{J}^r Q$ to $\mathcal{J}^r Q$. We may define a new tensor field $\tilde{S}$ of (1,1)-type $V\mathcal{J}^r Q$, by

$$\tilde{S} = S - C \otimes dt.$$

We define the adjoint $\tilde{S}^*$ of $\tilde{S}$, as the endomorphism of the exterior algebra $\Lambda(\mathcal{J}^r Q)$ of $\mathcal{J}^r Q$, locally given by

$$\tilde{S}^*(dt) = 0, \quad \tilde{S}^*(dq^i_{(0)}) = 0, \quad \tilde{S}^*(dq^i_{(h)}) = h\theta^i_{(h-1)}; \quad h = 1, 2, \dots, r$$

where

$$\theta^i_{(h)} = dq^i_{(h)} - q^i_{(h+1)} dt; \quad h = 0, 1, \dots, (r-1).$$

A vector field ξ on $\mathcal{J}^r Q$ is a semispray iff $S\xi = C$ and $\tilde{S}\xi = 0$.

Remark 1. It is not hard to see that a vector field ξ on $\mathcal{J}^r Q$ is a semispray iff $\theta^i_{(h)}(\xi) = 0$; $h = 0, 1, \dots, (r-1)$; $dt(\xi) = 1$.

In such a case ξ is locally given by

$$\xi = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}} + \xi^i \frac{\partial}{\partial q^i_{(r)}}. \tag{1.1}$$

Let now c be a global section of the affine bundle $\mathcal{J}^{r+1} Q \to \mathcal{J}^r Q$. One may construct a semispray ξ_c, given by

$$\xi_c = C^* \circ d_T. \tag{1.2}$$

Let $c(t, q^i_{(h)}) = (t, q^i_{(h)}, c^i)$; $0 \le h \le r$ be the local representation of c in a natural fibred chart. The semispray ξ_c is given by

$$\xi_c = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}} + c^i \frac{\partial}{\partial q^i_{(r)}}. \tag{1.3}$$

It follows then:

Proposition 1.1. *Let c be a section of the affine bundle $\mathcal{J}^{r+1}Q \to \mathcal{J}^r Q$ and ξ_c the spray given by (1.3), then we have:*

$$S\xi_c = C; \quad \tilde{S}\xi_c = 0; \quad \tilde{S} \circ L_{\xi_c}\tilde{S} = -\tilde{S} \circ \tilde{C}$$
$$(L_{\xi_c}\tilde{S} - rI)(L_{\xi_c}\tilde{S} + \tilde{C}) = 0.$$

Here L_{ξ_c} is the Lie derivative.

2. Semisprays and dynamical connections

The tensor fields S and $\tilde{S}$ on $\mathcal{J}^r Q$ permit us to give a characterization of a kind of connections for the fibration $\pi_0^r : \mathcal{J}^r Q \to Q$.

Definition 2.1. By a dynamical connection on $\mathcal{J}^r Q$ we mean a tensor field Γ of (1,1)-type on $\mathcal{J}^r Q$ satisfying

$$S\Gamma = \tilde{S}\Gamma = \tilde{S}; \quad \Gamma\tilde{S} = -\tilde{S}; \quad \Gamma S = -S. \tag{2.1}$$

By a straightforward computation we deduce from (2.1) that the local expression of Γ are:

$$\begin{cases} \Gamma\left(\dfrac{\partial}{\partial t}\right) = -\displaystyle\sum_{h=0}^{r-1} q^i_{(h+1)}\frac{\partial}{\partial q^i_{(h)}} + \Gamma^i_{(r)}\frac{\partial}{\partial q^i_{(r)}} \\ \Gamma\left(\dfrac{\partial}{\partial q^i_{(m)}}\right) = \dfrac{\partial}{\partial q^i_{(m)}} + \Gamma^{j(m)}_{i(r)}\dfrac{\partial}{\partial q^i_{(r)}} \qquad 0 \le m \le (r-1) \\ \Gamma\left(\dfrac{\partial}{\partial q^i_{(r)}}\right) = -\dfrac{\partial}{\partial q^i_{(r)}}. \end{cases} \tag{2.2}$$

The functions $\Gamma^i_{(r)} = \Gamma^i_{(r)}(t, q^i_{(h)})$; $\Gamma^{j(m)}_{i(r)} = \Gamma^{j(m)}_{i(r)}(t, q^i_{(h)})$ will be called the components of the connection Γ. From (2.2) we easily deduce that

$$\Gamma^3 - \Gamma = 0 \text{ and } \operatorname{rank}(\Gamma) = (r+1)n.$$

This type of polynomial structure is called $f(3,-1)$-structure in the literature [4]. Now, we can associate to Γ two canonical operators ℓ and m given by: $\ell = \Gamma^2$; $m = -\Gamma^2 + I$.
Then we have:

$$\ell^2 = \ell; \quad m^2 = m; \quad \ell m = m\ell = 0; \quad \ell + m = I \tag{2.3}$$

where ℓ and m are complementary projectors. From (2.3) we deduce that ℓ and m are locally given by:
(2.4)

$$\ell\left(\frac{\partial}{\partial t}\right) = -\sum_{h=0}^{r-1} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}} - \left(\Gamma^i_{(h)} + \sum_{h=0}^{r-1} q^i_{(h+1)} \Gamma^{j(h)}_{i(r)}\right) \frac{\partial}{\partial q^j_{(r)}}$$

$$\ell\left(\frac{\partial}{\partial q^i_{(k)}}\right) = \frac{\partial}{\partial q^i_{(k)}}; \quad m\left(\frac{\partial}{\partial q^i_{(k)}}\right) = \frac{\partial}{\partial q^i_{(k)}}; \qquad k = 0, 1, \ldots, r$$

$$m\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}} + \left(\Gamma^j_{(r)} + \sum_{h=0}^{r-1} q^i_{(h+1)} \Gamma^{j(h)}_{i(r)}\right) \frac{\partial}{\partial q^j_{(r)}}.$$

If we put $\mathcal{L} = \mathrm{Im}\ell$, $\mathcal{M} = \mathrm{Im}m$, then we have that $\mathcal{L}$ and $\mathcal{M}$ are complementary distributions on $\mathcal{J}^r Q$, that is

$$T(\mathcal{J}^r Q) = \mathcal{M} \oplus \mathcal{L}.$$

From (2.4) we deduce that $\mathcal{L}$ is $(r+1)n$-dimensional and is locally spanned by $\left\{\frac{\partial}{\partial q^i_{(k)}}\right\}$, $k = 0, 1, \ldots, r$; $\mathcal{M}$ is one-dimensional, and globally spanned by the vector field $\xi = m\left(\frac{\partial}{\partial t}\right)$. Taking into account the local expression of ξ we deduce that ξ is a semispray which will be called the canonical semispray associated to the dynamical connection Γ. Furthermore, we have $\Gamma^2\ell = \ell$ and $\Gamma m = 0$. Thus Γ acts on $\mathcal{L}$ as an almost product structure and trivially on $\mathcal{M}$. Since $\mathcal{M} = \mathrm{Ker}\,\Gamma$, Γ is said to be an $f(3,-1)$-structure on $\mathcal{J}^r Q$ of rank $(r+1)n$ and with parallelizable kernel. Moreover $\Gamma/\mathcal{L}$ has the eigenvalues $+1$ and -1. From (2.2) the eigenspaces corresponding to the eigenvalue $+1$ are the vertical subspaces V^0_z, $z \in \mathcal{J}^r Q$. Thus V is a distribution given by $z \to V^0_z$. The eigenspace at $z \in \mathcal{J}^r Q$ corresponding to the eigenvalue $+1$ will be denoted by H_z and called the strong-horizontal subspace at z. We have a canonical decomposition

$$T_z(\mathcal{J}^r Q) = \mathcal{M}_z \oplus H_z \oplus V^0_z$$

and obviously

$$T(\mathcal{J}^r Q) = \mathcal{M} \oplus H \oplus V^0$$

where H is the distribution $z \to H_z$.
Let us put $H'_z = \mathcal{M}_z \oplus H_z$; H'_z will be called the weak horizontal subspace at z. Then we have the following decompositions:

$$T_z(\mathcal{J}^r Q) = H'_z \oplus V^0_z, \qquad z \in \mathcal{J}^r Q$$

and

$$T(\mathcal{J}^r Q) = H' \oplus V^0 \tag{2.5}$$

where $H' : z \to H'_z$ is the corresponding distribution.

We notice that $\mathcal{L}$, $\mathcal{M}$, H and H' may be considered as vector bundles over $\mathcal{J}^r Q$; the bundles H and H' will be called strong and weak-horizontal bundles, respectively.

A vector field X on $\mathcal{J}^r Q$ which belongs to H (resp. H') will be called a strong (resp. weak) horizontal vector field. From (2.5) we have that the projection $\pi^r_{r-1} : \mathcal{J}^r Q \to \mathcal{J}^{r-1} Q$ induces an isomorphism

$$\pi^r_{0*} : H'_z \to T_{\pi^r_{r-1}(z)}(Q), \qquad z \in \mathcal{J}^r Q.$$

Then, if X is a vector field on $\mathcal{J}^{r-1} Q$, there exists an unique vector field $X^{H'}$ on $\mathcal{J}^r Q$ which is weak-horizontal and projects to X. The projection of $X^{H'}$ to H will be denoted by X^H.

From (2.2), by a straightforward computation we obtain

$$\text{(2.6)} \quad \begin{cases} \left(\dfrac{\partial}{\partial t}\right)^{H'} = \dfrac{\partial}{\partial t} + \left(\Gamma^j_{(r)} + \dfrac{1}{2}\displaystyle\sum_{h=0}^{r-1} q^i_{(h+1)} \Gamma^{j(h)}_{i(r)}\right) \dfrac{\partial}{\partial q^j_{(r)}} \\ \left(\dfrac{\partial}{\partial q^i_{(k)}}\right)^{H'} = \dfrac{\partial}{\partial q^i_{(k)}} + \dfrac{1}{2}\Gamma^{j(k)}_{i(r)} \dfrac{\partial}{\partial q^j_{(r)}} \qquad k = 0, 1, \dots, (r-1). \end{cases}$$

Then, if we put $H^{(k)}_i = \left(\dfrac{\partial}{\partial q^i_{(k)}}\right)^{H'}$ and $V^{(r)}_i = \dfrac{\partial}{\partial q^i_{(r)}}$, one deduces that $\left\{\xi, H^{(k)}_i, V^{(r)}_i\right\}$ is a local basis of vector fields on $\mathcal{J}^r Q$. In fact $\mathcal{M} = \langle \xi \rangle$, $H = \langle H^{(k)}_i \rangle$, $V = \langle V^{(r)}_i \rangle$ and $\left\{\xi, H^{(k)}_i, V^{(r)}_i\right\}$ is called an adapted basis of the $f(3,-1)$-structure Γ. In terms of $\left\{\xi, H^{(k)}_i, V^{(r)}_i\right\}$ (2.6) becomes

$$\left(\frac{\partial}{\partial t}\right)^{H'} = \xi - \sum_{k=0}^{r-1} q^i_{(k+1)} H^{(k)}_i; \quad \left(\frac{\partial}{\partial q^i_{(k)}}\right)^{H'} = H^{(k)}_i; \qquad k = 0, 1, \dots, (r-1).$$

Therefore, we obtain

$$\left(\frac{\partial}{\partial t}\right)^{H} = -\sum_{k=0}^{r-1} q^i_{(k+1)} H^{(k)}_i; \quad \left(\frac{\partial}{\partial q^i_{(k)}}\right)^{H} = H^{(k)}_i; \qquad k = 0, 1, \dots, (r-1).$$

If $X = \eta\dfrac{\partial}{\partial t} + \sum\limits_{h=0}^{r-1} X^i_{(h)}\dfrac{\partial}{\partial q^i_{(h)}}$ is a vector field on $\mathcal{J}^{r-1}Q$ we have

$$X^H = \sum_{h=0}^{r-1}(X^i_{(h)} - \eta q^i_{()})H^{(h)}_i.$$

Finally, we notice that the dual local basis of 1-forms of the adapted basis is given by $(dt, \theta^i_{(h)}, \psi^i)$, where

$$\theta^i_{(h)} = dq^i_{(h)} - q^i_{(h+1)}dt; \qquad h = 0, 1, \dots, (r-1) \text{ and}$$
$$\psi^i = -\left(\Gamma^i_{(r)} + \frac{1}{2}\sum_{h=0}^{r-1} q^i_{(h+1)}\Gamma^{j(h)}_{i(r)}\right)dt - \frac{1}{2}\sum_{h=0}^{r-1}\Gamma^{i(h)}_{j(r)}dq^j_{(h)} + dq^i_{(r)}.$$

Let ξ be a semispray of $\mathcal{J}^rQ$ and we suppose that ξ is locally expressed by (1.1). Then a simple computation in local coordinates shows that we have:

$$\text{(2.7)}\qquad \begin{cases} \left[\xi, \dfrac{\partial}{\partial t}\right] = -\dfrac{\partial \xi^i}{\partial t}\dfrac{\partial}{\partial q^i_{(r)}} \\ \left[\xi, \dfrac{\partial}{\partial q^i_{(h)}}\right] = -\dfrac{\partial \xi^j}{\partial q^i_{(h)}}\dfrac{\partial}{\partial q^j_{(r)}} - \dfrac{\partial}{\partial q^i_{(h-1)}} \qquad h = 1, 2, \dots, r \\ \left[\xi, \dfrac{\partial}{\partial q^i_{(0)}}\right] = -\dfrac{\partial \xi^j}{\partial q^i_{(0)}}\dfrac{\partial}{\partial q^j_{(r)}}. \end{cases}$$

Proposition 2.1. *Let $\Gamma = -L_\xi\tilde{S}$. Then Γ is a dynamical connection on $\mathcal{J}^rQ$, whose associated semispray is precisely ξ.*

Proof. In fact from (2.7) we have:

$$\text{(2.8)}\qquad \begin{cases} \Gamma\left(\dfrac{\partial}{\partial t}\right) = -\displaystyle\sum_{h=0}^{r-1} q^i_{(h+1)}\dfrac{\partial}{\partial q^i_{(h)}} \\ \qquad\qquad - \left(\displaystyle\sum_{h=0}^{r-1}(h+1)q^i_{(h+1)}\dfrac{\partial \xi^j}{\partial q^i_{(h+1)}} - r\xi^i\right)\dfrac{\partial}{\partial q^i_{(r)}} \\ \Gamma\left(\dfrac{\partial}{\partial q^i_{(k)}}\right) = \dfrac{\partial}{\partial q^i_{(k)}} + \dfrac{\partial \xi^j}{\partial q^i_{(k+1)}}\dfrac{\partial}{\partial q^j_{(r)}}; \quad k = 0, 1, 2, \dots, (r-1) \\ \Gamma\left(\dfrac{\partial}{\partial q^i_{(r)}}\right) = -r\dfrac{\partial}{\partial q^i_{(r)}}. \end{cases}$$

Now, from (2.8) we easily deduce that Γ is a dynamical connection on $\mathcal{J}^r Q$. Furthermore, taking into account (2.4), we have that the associated semispray to Γ is precisely ξ.

Let Γ be a dynamical connection on $\mathcal{J}^r Q$. A curve $s : X \to Q$ is called a path of Γ if the canonical prolongation $j^r s$ of s to $\mathcal{J}^r Q$ is a weak-horizontal curve.

If $s : X \to Q$ is locally given by $t \to (t, q^i(t))$, then we have $j^r s(t) = (t, q^i_{(h)}(t))$; $0 \leq h \leq r$.

Hence

$$\widehat{\dot{j^r s(t)}} = \frac{\partial}{\partial t} + \sum_{h=1}^{r+1} \frac{d^h q^i}{dt^h} \frac{\partial}{\partial q^i_{(h-1)}}.$$

Therefore s is a path of Γ if and only if $\psi^i \left(\widehat{\dot{j^r s(t)}}\right) = 0$; $i = 1, 2, \ldots, n$, that is, s satisfies the following system of differential equations:

$$\frac{d^{r+1} q^i}{dt^{r+1}} = \Gamma^i_{(r)} + \sum_{h=0}^{r-1} \Gamma^{i(h)}_{j(r)} \frac{d^{h+1} q^j}{dt^{h+1}}. \tag{2.9}$$

Let ξ be the associated semispray of Γ. Then ξ is locally given by:

$$\xi = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}} + \xi^i \frac{\partial}{\partial q^i_{(r)}}$$

where:

$$\xi^i = \Gamma^i_{(r)} + \sum_{h=0}^{r-1} q^j_{(h+1)} \Gamma^{i(h)}_{j(r)}; \qquad 1 \leq i \leq n.$$

From (2.9) it is clear that the paths of Γ and ξ satisfy the same system of differential equations. Then we have:

Proposition 2.2. *A dynamical connection and its associated semispray on $\mathcal{J}^r Q$ have the same paths.*

3. The generalized Euler-Lagrange operator

Let $c : \mathcal{J}^r Q \to \mathcal{J}^{r+1} Q$ be a global section and ξ_c given by (1.2). The generalized Euler Lagrange operator associated to ξ_c is the R-linear operator on 1-forms $\mathcal{E}_{\xi_c}$ defined by

$$\mathcal{E}_{\xi_c} = -\xi_c \otimes dt + \sum_{h=0}^{r} (-1)^h \frac{1}{h!} L^h_{\xi_c} \cdot \tilde{S}^h. \tag{3.1}$$

Proposition 3.1. *The Euler-Lagrange operator satisfies*

$$\tilde{S} \circ \mathcal{E}_{\xi_c} = 0.$$

If $\omega \in \Lambda^1(\mathcal{J}^r Q)$ *is given by:*

$$\omega = \alpha dt + \sum_{h=0}^{r} \omega_i^{(h)} dq^i \tag{3.2}$$

then

$$\mathcal{E}_\Gamma(\omega) = \left(\sum_{h=0}^{r} (-1)^r \xi_c^h (\omega_i^{(h)}) \right) \theta^i_{(0)}. \tag{3.3}$$

To each semispray ξ_c we now associate a set of 1-forms $X^*_{\xi_c} = Ker \mathcal{E}_{\xi_0^*}$. The set $X^*_{\xi_c}$ is in fact a vector space over R, by the R-linearity of $\mathcal{E}_{\xi_c}$. Its elements satisfy the relation:

$$\omega_i^{(0)} - \xi_c(\omega_i^{(1)}) + \xi_c^2(\omega_i^{(2)}) - \ldots + (-1)^r \xi_c^r(\omega_i^{(r)}) = 0. \tag{3.4}$$

Furthermore, we define an R-linear operator σ_{ξ_c} of 1-forms on $\mathcal{J}^r Q$, called the generalized Cartan operator, by:

$$\sigma_{\xi_c} = \sum_{h=0}^{r-1} (-1)^h \frac{1}{(h+1)!} L^h_{\xi_c} \circ \tilde{S}^{(h+1)}. \tag{3.5}$$

It follows from this definition that:

$$\mathcal{E}_{\xi_c} \omega = \omega - i_{\xi_c} \omega dt - L_{\xi_c} \circ \sigma_{\xi_c} \omega. \tag{3.6}$$

Let ξ_c be a semispray given by (1.2) and $L \in \mathcal{F}(\mathcal{J}^r Q)$. The Poincaré-Cartan 1-form θ_{L,ξ_c} is given by:

$$\theta_{L,\xi_c} = L dt + \sigma_{\xi_c}(dL). \tag{3.7}$$

We say that a semispray ξ_c is Lagrangian if there exists $L \in \mathcal{F}(\mathcal{J}^r Q)$ such that $dL \in X^*_{\xi_c}$.

Proposition 3.2. ξ_c *is Lagrangian iff there exists* $L \in \mathcal{F}(\mathcal{J}^r Q)$ *such that:*

$$i_{\xi_c} \omega_L = 0 \tag{3.8}$$

where $\omega_L = -d\theta_{L,\xi_c}$. *We call* ξ_c *the Lagrange vector for* L.

We now describe how the usual formulation of higher-order dynamics fits into to framework described above. It will be recalled that the Lagrangian function of $\mathcal{J}^r Q$ leads to Euler-Lagrange equations which are

$2r$-order differential equations. We must therefore consider the $(2r-1)$-order jet-prolongation $\mathcal{J}^{2r-1}Q$ and functions on it of the form $\pi_r^{2r-1*}L$, where L is a function on $\mathcal{J}^rQ$.

It follows that, if ξ_c is a $2r$-order differential equation field which is Lagrangian, with Lagrangian function L on $\mathcal{J}^rQ$, then :

$$\frac{\partial L}{\partial q^i_{(0)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^i_{(0)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q^i_{(0)}}\right) + \ldots + (-1)^r \frac{d^r}{dt^r}\left(\frac{\partial L}{\partial q^i_{(0)}}\right) = 0 \tag{3.9}$$

along any integral curve of ξ_c.

Proposition 3.3. *Let L be a non-autonomous regular Lagrangian on $\mathcal{J}^rQ$, and let ξ_c be a Lagrange vector field for L. Then there exists a dynamical connection Γ on $\mathcal{J}^{2r-1}Q$ whose paths are the solutions of the equations. This connection is given by $\Gamma = -L_{\xi_c}\tilde{S}$.*

References

[1] M. Crampin and W. Sarlet and F. Cantrijn, Higher-order differential equations and higher-order Lagrangian mechanics, *Math. Proc. Camb. Phil. Soc.* **99** (1986), 565.

[2] M. Ferraris and M. Francaviglia, On the globalization of Lagrangian and Hamiltonian formalisms in higher order mechanics, *Atti della Accademia delle Scienze di Torino* **117** (1983).

[3] M. De Leon and P. Rodrigues, Dynamical connections and non-autonomous Lagrangian systems, *Annales Faculté des Sciences de Toulouse* **IX,** no. 2 (1988).

[4] K. Yano and S. Ishihara, Tangent and cotangent bundles, *Pure and Appl. Math. Ser.* **16.** Marcel Dekker N. Y. (1973).

D. OPRIS
SEMINAR OF GEOMETRY AND TOPOLOGY
UNIVERSITY OF TIMIŞOARA
ROUMANIA

F.C. KLEPP
DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF TIMIŞOARA
ROUMANIA

(Received August 29, 1991; revised March 10, 1992)