

Pethő's cubics

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*This paper is dedicated to Kálmán Győry,
for the occasion of his 60th birthday*

Abstract. We compute all the solutions of the family of cubic Thue equations

$$\Phi_n(x, y) = x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$$

for all rational integers n .

1. Introduction

We continue the study of a non-Galois family of cubic Thue equations $\Phi_n(x, y) = 1$ which was initiated in a joint paper with N. TZANAKIS [MT]. The associated fields $\mathbb{Q}(\theta_n)$, where $\Phi_n(\theta_n, 1) = 0$, are totally real.

The family of cubics we consider is

$$(1) \quad \Phi_n(x, y) = x^3 - nx^2y - (n+1)xy^2 - y^3.$$

Notice that the transformation $(x, y) \mapsto (-y, -x)$ defines a one-to-one correspondence between the solutions of the equations $\Phi_n(X, Y) = 1$ and $\Phi_{-n-1}(X, Y) = 1$, thus we consider only the case $n \geq 0$.

Note also that each equation $\Phi_n(x, y) = 1$ has the solutions $(x, y) = (1, 0), (0, -1), (1, -1), (-n-1, -1), (1, -n)$. This gives five “trivial solutions” for $n \neq 0, 1$ and four ones otherwise. To simplify we solve the

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equations for $0 \leq n \leq 19$ using Kant, this shows that in this range the equation has only trivial solutions except for $n = 0$ where there is the extra solution $(x, y) = (4, 3)$, for $n = 3$ (then the discriminant is 49) where there are the four non-trivial solutions $(-5, 14)$, $(-2, 3)$, $(-1, 2)$ and $(9, -13)$ and for $n = 4$ (then the discriminant is 257) where there is the non-trivial solution $(7, -9)$. From now on we always suppose $n \geq 20$, without referring explicitly to this assumption.

According to a conjecture of A. PETHŐ [P] based on extensive computations, for any irreducible cubic form $\Phi_n(x, y) \in \mathbb{Z}[x, y]$ with positive discriminant $\neq 49, 81, 148, 257, 361$, the equation $\Phi_n(x, y) = 1$ has at most five solutions. In [MT], it is proved that, indeed, the above mentioned five solutions are the only solutions of the equation

$$(2) \quad x^3 - nx^2y - (n+1)xy^2 - y^3 = 1,$$

if $n \geq 3.67 \times 10^{32}$, in accordance to Pethő's conjecture. (We chose the title of this paper because this family gives the maximum number of solutions known for a family of cubics.) Here we prove this result for all $n \geq 5$:

Theorem. *If $n \geq 5$, then the only solutions of the diophantine equation*

$$x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$$

are

$$(x, y) = (1, 0), (0, -1), (1, -1), (-n-1, -1), (1, -n).$$

We give a sketch of the method, which contains several steps. We work in number fields K attached to the Thue equation, depending on the parameter n . We know explicitly a fundamental system $\{\xi, 1 + \xi\}$, for the units of K ; and we notice that a solution (x, y) of the Thue equation satisfies $x + y\xi = \xi^a(1 + \xi)^b$.

It is understood that all estimates and bounds referred to below are explicit and contain the parameter n , except if they are explicitly characterized as “numerical”. The plan is the following.

1. Estimate the regulator R of K
2. Find an upper bound for $A := \max\{|a|, |b|\}$, in terms of R and $\log |y|$.
3. Obtain an upper bound for the linear form $|\Lambda|$ in three logarithms obtained by Siegel's formula, of the form $|\Lambda| = O(|y|^{-3})$.

Combine the results of steps 1, 2, 3 to find an upper bound for $|\Lambda|$ in terms of A .

5. Find a lower bound for A : this is a fundamental step, and there is no systematic way to get it.
6. Combine the results of steps 4 and 5 to obtain a negative upper bound for $\log |\Lambda|$.
7. Transform Λ into a homogeneous linear form in two logarithms in order that the sharp result of Laurent–Mignotte–Nesterenko can be applied to give a good negative lower bound for $\log |\Lambda|$.
8. Combine the results of steps 6, 7 to obtain a *numerical* upper bound for n , say $n \leq N$.
9. View Λ , again, as a homogeneous linear form in three logarithms and apply Waldschmidt's result in order to obtain a negative lower bound for $\log |\Lambda|$, containing A .
10. Combine the results of steps 4, 9 to obtain a *numerical* upper bound for A .
11. Apply a lemma *à la* Baker–Davenport, in which the bound for A , obtained in step 10, is necessary, to treat the values of $n \leq N$, the bound found in step 8.

2. Preliminaries

We work in the field $K = Q(\xi)$, where $\xi^3 - n\xi^2 - (n+1)\xi - 1 = 0$ (clearly $\xi = \xi_n$ and $K = K_n$ depend on n). The equation $x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$ implies that $x - y\xi$ is a unit of K .

The discriminant of ξ is $n^4 + 2n^3 - 5n^2 - 6n - 23 = (n^2 + n - 3)^2 - 32$, hence it is positive for $n \geq 3$ and it is a square only if $n = 3$, hence K is not Galois for $n > 3$. For $n \geq 4$ we know two fundamental units in K : Put $\xi = \lambda^{-1} - 1$. Then $K = Q(\lambda)$ and $\lambda^3 - (n+2)\lambda^2 + (n+3)\lambda - 1 = 0$, therefore, by E. THOMAS' paper [T1], a pair of fundamental units is λ , $\lambda - 1$, i.e. $1/(1+\xi)$ and $(-\xi)/(1+\xi)$. From this it follows that ξ , $\xi + 1$ is a pair of fundamental units of K . Then, $x - y\xi = \pm \xi^a(1+\xi)^b$ for some a , $b \in \mathbb{Z}$. Since the norms of ξ and $1+\xi$ are $+1$, the minus sign is excluded and

$$x - y\xi = \xi^a(1 + \xi)^b.$$

Put

$$F(X) = F_n(X) = X^3 - nX^2 - (n+1)X - 1.$$

We can have good estimates of the roots of F by appropriate substitutions. Since $F(n+1) = -1$ and also $F(n+1+n^{-2}) = 3n^{-1} + 2n^{-2} + 2n^{-3} + 3n^{-4} + n^{-6} > 0$, the polynomial F has a root, say ξ_1 , with

$$(3)_1 \quad n+1 < \xi_1 < n+1+n^{-2}.$$

Similarly, sign changes of the polynomial F show that

$$(3)_2 \quad -1 + \frac{1}{n+1} < \xi_2 < -1 + \frac{1}{n+1} + \frac{1}{(n+1)^2}$$

and

$$(3)_3 \quad -\frac{1}{n} - \frac{1}{n^3} < \xi_3 < -\frac{1}{n}.$$

We shall often use the simpler following estimates: the roots of ξ_1, ξ_2, ξ_3 of F satisfy:

$$(3) \quad \begin{aligned} n+1 < \xi_1 < n+1 + \frac{1}{n^2}, \\ -\frac{n}{n+1} < \xi_2 < -\frac{n-1}{n}, \quad -\frac{1}{n-1} < \xi_3 < -\frac{1}{n}, \end{aligned}$$

But, more precise estimates will also be necessary. We use the Lagrange's method to compute the beginning of the continued fraction expansion of the ξ 's.

R1) *Approximate value of ξ_1*

By the change of variable $X = n+1 + Y^{-1}$, the polynomial F is transformed into $g(Y) = -Y^3 + (n^2 + 3n + 2)Y^2 + (2n + 3)Y + 1$. Since $g(n^2 + 3n + 2) = 2n^3 + 9n^2 + 13n + 7 > 0$, and $g(n^2 + 3n + 3) = -n^4 - 4n^3 - 6n^2 - 3n + 1 < 0$, we have

$$n+1 + \frac{1}{n^2 + 3n + 3} < \xi_1 < n+1 + \frac{1}{n^2 + 3n + 2},$$

thus the beginning of the continued fraction expansion of ξ_1 is

$$\xi_1 = [n+1; n^2 + 3n + 2, \dots].$$

R2) *Approximate value of ξ_2*

By the successive changes of variables $X = -1 + Y^{-1}$, $Y = n + Z^{-1}$ and $Z = 1 + T^{-1}$ we get the continued fraction expansion

$$\xi_2 = [-1; n, 1, \lfloor (n-2)/2 \rfloor, \dots].$$

Which shows that

$$-1 + \frac{1}{n+1-\frac{2}{n-3}} < \xi_2 < -1 + \frac{1}{n+1-\frac{2}{n}}.$$

R3) *Approximate value of ξ_3*

By a similar study we see that

$$\xi_3 = -[0; n-1, 1, n^2-n-2, \dots],$$

hence

$$-\frac{1}{n-\frac{1}{n^2-n-1}} < \xi_3 < -\frac{1}{n-\frac{1}{n^2-n}}.$$

Notice also the formulae

$$\Phi_n(x, n-1) = x^3 - (n^2 - n)x^2 - (n^3 - n^2 - n + 1)x - (n^3 - 3n^2 + 3n - 1),$$

$$\Phi_n(x, n) = x^3 - n^2x^2 - (n^3 + n^2)x - n^3,$$

$$\Phi_n(x, n+1) = x^3 - (n^2 + n)x^2 - (n^3 + 3n^2 + 3n + 1)x - (n^3 + 3n^2 + 3n + 1).$$

We make a very elementary study of the solutions of equation (2):

- If $y = 0$ then, clearly, $x = 0$.
- If $|y| = 1$, consider first the case $y = 1$, then $\Phi_n(x, y) = x^3 - nx^2 - (n+1)x - 1 = g(x)$, say. It is easy to verify that $g(x) = -1$ iff $x \in \{-1, 0, n+1\}$ and that $|g(x)| > 1$ for all other $x \in Z$, hence $\Phi_n(x, 1) \neq 1$ for any $x \in Z$. If $y = -1$ then since $\Phi_n(x, -y) = -\Phi(-x, y)$, we have $\Phi_n(x, -y) = 1$ iff $x \in \{1, 0, -(n+1)\}$, showing that in this case solutions (x, y) are the “trivial ones” $(0, -1)$, $(1, -1)$ and $(-n-1, -1)$.
- If $|y| = 2$, consider first the case $y = 2$, then $\Phi_n(x, y) = x^3 - 2nx^2 - 4(n+1)x - 8 = h(x)$, say. And it is easy to verify that $|h(x)| \geq 8$ for $x \neq -1$, whereas $h(-1) = 2n - 5$. Thus $\Phi_n(x, 2) = 1$ only when $n = 3$ and $x = -1$. Moreover, using again the formula $\Phi_n(x, -y) = -\Phi(-x, y)$,

we see that the diophantine equation $\Phi_n(x, -2) = 1$ has no solution for $n \geq 3$. Thus we may now suppose that $|y| \geq 3$.

From formula (2), we have

$$(x - \xi_1 y)(x - \xi_2 y)(x - \xi_3 y) = 1.$$

Let i be the index such that

$$|x - \xi_i y| = \min_{1 \leq j \leq 3} |x - \xi_j y|,$$

then $|x - \xi_i y| < 1$ and, by the estimate (3) for the roots of F , $\xi_1 - \xi_k > n + 1$ for $k \neq 1$, thus

$$(j \neq i) \ \& \ (1 \in \{i, j\})$$

$$\Rightarrow |x - \xi_j y| \geq |\xi_j - \xi_i| |y| - |x - \xi_i y| \geq (n + 1)|y| - 1 > (n + 2/3)|y|.$$

Hence, $|x - \xi_i y|^2 (n + 2/3)|y| < 1$, in other words

$$|x - \xi_i y| < ((n + 2/3)|y|)^{-1/2} \leq 1/\sqrt{62},$$

in particular i is indeed unique.

For $n \geq 20$, by (3),

$$|\xi_2 - \xi_3| > 1 - \frac{1}{n+1} - \frac{1}{n-1} = 1 - \frac{2n}{n^2 - 1}$$

and by a previous computation $|x - \xi_i y| < 1/\sqrt{62}$, thus

$$\prod_{j \neq i} |x - \xi_j y| > n \left(1 - \frac{40}{399} - \frac{1}{2\sqrt{62}} \right) y^2 > 0.836 n y^2 > 16y^2$$

and

$$\left| \frac{x}{y} - \xi_i \right| < \frac{1}{0.836n|y|^3} < \frac{1}{16|y|^3}.$$

This short study proves that the rational number x/y is a principal convergent of ξ_i . Now, we have to consider the three cases $i = 1, 2, 3$.

$$\boxed{i = 1}$$

Then $\xi_1 = [n + 1; n^2 + 3n + 2, \dots]$ thus $|y| \geq n^2 + 3n + 2$.

$$\boxed{i = 2}$$

Then $\xi_2 = [-1; n, 1, \lfloor (n-2)/2 \rfloor, \dots]$ thus $|y| = n, n+1$ or $|y| \geq (n^2-3)/2$.

$$\boxed{i = 3}$$

Then $\xi_3 = -[0; n-1, 1, n^2-n-2, \dots]$ thus $|y| = n-1$, n or $|y| > n^2$.

If $y = \pm(n-1)$ then $x = \pm 1$ and it is easy to verify that no solution (x, y) with $|y| = n-1$ exists.

If $y = \pm n$ then $x = \pm(n-1)$ or $x = \mp 1$. In the second case we have $\Phi_n(1, -n) = 1$, finding the last “trivial solution” $(x, y) = (1, -n)$. While, in the first case a direct computation shows that there is no solution (x, y) with $|y| = n$.

If $y = \pm(n+1)$ then $x = \pm n$ and by direct computation we see that there is no solution (x, y) with $|y| = n+1$.

This elementary study shows that a non trivial solution (x, y) must satisfy $|y| \geq (n^2-3)/2$.

Put $\varepsilon_h = 1 + \xi_h$ for $h = 1, 2, 3$. From the formula $x - \xi y = \xi^a(1 + \xi)^b$, we get

$$x - \xi_j y = \xi_j^a \varepsilon_j^b, \quad 1 \leq j \leq 3,$$

which implies

$$(4) \quad y(\xi_{j+1} - \xi_j) = \xi_j^a \varepsilon_j^b - \xi_{j+1}^a \varepsilon_{j+1}^b,$$

where $\xi_{j+1} = \xi_{(j+1) \bmod 3}$, with some abuse of notation.

Now we want to estimate the exponents a and b in terms of y . Put $l_h = \log |\xi_h|$, $l'_h = \log |\varepsilon_h|$ and $\mu_h = \log |x - \xi_h y|$ for $h = 1, 2, 3$. Then the relations $x - \xi_h y = \xi_h^a \varepsilon_h^b$ can be written as $l_h a + l'_h b = \mu_h$, from which we get, for $j \neq k$,

$$(5) \quad a = \frac{l'_k \mu_j - l'_j \mu_k}{l'_k l_j - l'_j l_k}, \quad b = -\frac{l_k \mu_j - l_j \mu_k}{l'_k l_j - l'_j l_k}.$$

Put $R = l'_2 l_3 - l'_3 l_2$; using the obvious relations $l_1 + l_2 + l_3 = 0$ and $l'_1 + l'_2 + l'_3 = 0$ it is easy to verify that $R = l'_3 l_1 - l'_1 l_3 = l'_1 l_2 - l'_2 l_1$ and we shall see that R is positive.

From the estimates (3) one easily deduces that

$$\begin{aligned} \log(n+1) < l_1 < \log(n+2), \quad -\frac{1}{n-1} < l_2 < -\frac{1}{n+1}, \\ -\log n < l_3 < -\log(n-1), \end{aligned}$$

$$\begin{aligned} \log(n+2) < l'_1 < \log(n+3), \quad -\log(n+1) < l'_2 < -\log n, \\ -\frac{1}{n-2} < l'_3 < -\frac{1}{n}. \end{aligned}$$

To estimate the μ_h we can write, for $h \neq i$,

$$|(\xi_i - \xi_h)y| - \frac{1}{16y^2} \leq |x - \xi_h y| = |(\xi_i - \xi_h)y + x - \xi_i y| \leq |(\xi_i - \xi_h)y| + \frac{1}{16y^2},$$

since $|\xi_i - \xi_h| > 0.8$ we get

$$|(\xi_i - \xi_h)y| \left(1 - \frac{0.08}{|y|^3}\right) \leq |x - \xi_h y| \leq |(\xi_i - \xi_h)y| \left(1 + \frac{0.08}{|y|^3}\right).$$

This implies

$$(6) \quad \log |\xi_i - \xi_h| - \frac{0.1}{|y|^3} \leq \mu_h - \log |y| \leq \log |\xi_i - \xi_h| + \frac{0.1}{|y|^3}.$$

From the estimates of the roots of F it is easy to check the following inequalities

$$\begin{aligned} 1 - \frac{2n}{n^2 - 1} < \xi_3 - \xi_2 < 1 - \frac{2}{n+1}, \\ n+2 - \frac{1}{n} < \xi_1 - \xi_2 < n+2, \\ n+1 + \frac{1}{n} < \xi_1 - \xi_3 < n+1 + \frac{1}{n-2}. \end{aligned}$$

It will be very useful to have estimates for R . From the above estimates of the l_h and l'_h , we get

$$\log(n-1) \times \log n - \frac{1}{(n-1)(n-2)} < R < \log n \times \log(n+1).$$

Indeed, we have the simpler estimate

$$(7) \quad \log^2(n-1) < R < \log n \times \log(n+1).$$

Now we want to get estimates of a and b in terms of y . To simplify the notations, put

$$\eta = 0.1|y|^{-3} \quad \text{and} \quad z = \log |y|.$$

We have to distinguish the three cases $i = 1, 2$ and 3 .

$$\boxed{i = 1}$$

By (5),

$$a = \frac{l'_2\mu_3 - l'_3\mu_2}{R}, \quad b = \frac{l_3\mu_2 - l_2\mu_3}{R}.$$

Here,

$$\log(n+2-1/n) - \eta \leq \mu_2 - z \leq \log(n+2) + \eta,$$

$$\log(n+1) - \eta \leq \mu_3 - z \leq \log(n+2) + \eta.$$

Hence, a and b are negative and we have

$$l'_3\mu_2 - l'_2\mu_3 \leq |l'_2|\mu_3 \leq \log n \times (z + \log(n+2) + \eta),$$

$$l_2\mu_3 - l_3\mu_2 \leq |l_3|\mu_2 \leq \log n \times (z + \log(n+2) + \eta),$$

thus

$$A \leq \frac{\log n}{R} (z + \log(n+2) + \eta) \leq \frac{\log n}{\log^2(n-1)} (z + \log(n+2) + \eta),$$

where we have put

$$A = \max\{|a|, |b|\}.$$

This implies

$$(8)_1 \quad A \leq z \frac{\log n}{R} + 3.$$

$$\boxed{i = 2}$$

By (5),

$$a = \frac{l'_3\mu_1 - l'_1\mu_3}{R}, \quad b = \frac{l_1\mu_3 - l_3\mu_1}{R}.$$

Here,

$$\log(n + 2 - 1/n) - \eta \leq \mu_1 - z \leq \log(n + 2) + \eta,$$

$$\log\left(1 - \frac{2n}{n^2 - 1}\right) - \eta \leq \mu_3 - z \leq \eta.$$

In this case $a < 0 < b$ and we have

$$0 < l'_1\mu_3 - l'_3\mu_1 \leq \log(n + 3) \cdot (z + \eta) + \frac{1}{n - 2}(z + \log(n + 2) + \eta),$$

$$|l_3\mu_1 - l_1\mu_3| \leq \log n \cdot (z + \log(n + 2) + \eta) + \log(n + 2) \cdot (z + \eta),$$

thus

$$(8)_2 \quad A \leq \frac{2 \log(n + 2)}{R} (z + \tfrac{1}{2} \log(n + 2) + \eta).$$

$$\boxed{i = 3}$$

Here,

$$a = \frac{l'_1\mu_2 - l'_2\mu_1}{R}, \quad b = \frac{l_2\mu_1 - l_1\mu_2}{R}.$$

And,

$$\log(n + 2 - 1/n) - \eta \leq \mu_1 - z \leq \log(n + 2) + \eta,$$

$$\log\left(1 - \frac{2n}{n^2 - 1}\right) - \eta \leq \mu_2 - z \leq \eta.$$

In this case $b < 0 < a$ and we have

$$|l'_2\mu_1 - l'_1\mu_2| \leq \log(n + 1) \cdot (z + \log(n + 2) + \eta) + \log(n + 3) \cdot (z + \eta),$$

$$|l_2\mu_1 - l_1\mu_2| \leq \log(n + 2) \cdot (z + \eta) + \frac{1}{n - 1} \cdot (z + \log(n + 2) + \eta),$$

thus

$$(8)_3 \quad A \leq \frac{2 \log(n+3)}{R} \left(z + \frac{1}{2} \log(n+2) + \eta \right).$$

Comparing the inequalities $(8)_i$ we get the following conclusion:

$$(8) \quad A \leq \frac{2 \log(n+3)}{R} \left(z + \frac{1}{2} \log(n+2) + \eta \right).$$

3. A study of a linear form in three variables

In our case Siegel's identity is

$$(\xi_i - \xi_j) \xi_k^a \varepsilon_k^b + (\xi_j - \xi_k) \xi_i^a \varepsilon_i^b + (\xi_k - \xi_i) \xi_j^a \varepsilon_j^b = 0$$

which leads to the relation

$$\frac{(\xi_j - \xi_i) \xi_k^a \varepsilon_k^b}{(\xi_k - \xi_i) \xi_j^a \varepsilon_j^b} - 1 = \frac{(\xi_k - \xi_j) \xi_i^a \varepsilon_i^b}{(\xi_i - \xi_k) \xi_j^a \varepsilon_j^b}.$$

We choose $j = i+1$ and $k = i+2$, where these values are counted modulo 3, and consider the linear form of three logarithms

$$\Lambda = \log |\delta_i| + a \log |\xi_k / \xi_j| + b \log |\varepsilon_k / \varepsilon_j|,$$

where

$$\delta_i = \frac{\xi_{i+1} - \xi_i}{\xi_{i+2} - \xi_i}.$$

Then elementary computation using estimates of Section 2 show that

$$(9) \quad |\Lambda| < \frac{|\xi_k - \xi_j|}{|\xi_i - \xi_k|} \times \frac{1}{0.709n|y|^3} \times \frac{1.02}{|\xi_i - \xi_j|} < \frac{5}{n|y|^3}.$$

Using (8) and (7) this implies

$$(10) \quad \begin{aligned} \log |\Lambda| &< -\frac{3A}{2} \frac{R}{\log(n+2)} + \frac{3}{2} \log(n+3) + 2 \\ &< -\frac{3A \log^2(n-1)}{2 \log(n+3)} + \frac{3}{2} \log(n+3) + 2. \end{aligned}$$

Now, we have to find upper bounds for the heights of the algebraic numbers which appear in Λ .

M1) *Measure of δ*

We have $\delta = \frac{\xi_3 - \xi_2}{\xi_1 - \xi_2}$, this number is reciprocal. The conjugates of modulus > 1 correspond to a numerator which contains the largest of the conjugates of ξ , this shows that the measure of δ is given by the formula

$$\begin{aligned} M(\delta) &= \left| \frac{\xi_1 - \xi_3}{\xi_2 - \xi_3} \right| \cdot \left| \frac{\xi_2 - \xi_1}{\xi_3 - \xi_1} \right| \cdot \left| \frac{\xi_1 - \xi_2}{\xi_3 - \xi_2} \right| \cdot ((\xi_1 - \xi_2)(\xi_2 - \xi_3)(\xi_3 - \xi_1))^2 \\ &= |\xi_1 - \xi_2|^4 |\xi_1 - \xi_3|^2. \end{aligned}$$

Thus

$$M(\delta) < (n+2)^6.$$

M2) *Measure of ξ_1/ξ_2*

This number is a unit and it is also reciprocal. Its conjugates of modulus > 1 correspond to a numerator which contains the largest of the conjugates of ξ , or to the denominator equal to the smallest conjugate. Thus

$$\begin{aligned} M(\xi_1/\xi_2) &= |(\xi_1/\xi_2) \cdot (\xi_1/\xi_3) \cdot (\xi_2/\xi_3)| \\ &= (\xi_1/\xi_3)^2 < \left(\frac{n+1+1/n}{1/n} \right)^2 < (n+2)^4. \end{aligned}$$

M3) *Measure of $\varepsilon_1/\varepsilon_2$*

The same arguments than for the study of δ apply and show that the measure of $\varepsilon_1/\varepsilon_2$ satisfies

$$M(\varepsilon_1/\varepsilon_2) = \left| \frac{\varepsilon_1}{\varepsilon_2} \right| \cdot \left| \frac{\varepsilon_1}{\varepsilon_3} \right| \cdot \left| \frac{\varepsilon_3}{\varepsilon_2} \right| \cdot (\varepsilon_1 \varepsilon_2 \varepsilon_3)^2 = \varepsilon_1^4 \varepsilon_3^2.$$

This easily leads to the estimate

$$M(\varepsilon_1/\varepsilon_2) < (n+2)^4.$$

We quote the result of [LMN] that we shall use three times.

Proposition 1. *Let α_1, α_2 be nonzero algebraic numbers, and let $\log \alpha_1$ and $\log \alpha_2$ be any determinations of their logarithms. Consider the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers. We suppose that $|\alpha_1|$ and $|\alpha_2|$ are ≥ 1 . Put

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] / [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

Let a_1, a_2, h, ρ be positive real numbers, with $\rho > 1$. Put $\lambda = \log \rho$ and suppose that

- (i) $h \geq \max \left\{ \frac{D}{2}, 5\lambda, D \left(\log \left(\frac{b_1}{a_2} + \frac{b_2}{a_1} \right) + \log \lambda + 2.1 \right) \right\},$
- (ii) $a_i \geq \max \left\{ 2, 2\lambda, \rho |\log \alpha_i| - \log |\alpha_i| + 2Dh(\alpha_i) \right\}, \quad (i = 1, 2).$

When α_1 and α_2 are multiplicatively independent, we have

$$\begin{aligned} \log |\Lambda| &\geq -\frac{\lambda a_1 a_2}{9} \left(\frac{4h}{\lambda^2} + \frac{4}{\lambda} + \frac{1}{h} \right)^2 - \frac{2\lambda}{3} (a_1 + a_2) \left(\frac{4h}{\lambda^2} + \frac{4}{\lambda} + \frac{1}{h} \right) \\ \text{(iii)} \quad &-\frac{16\sqrt{2a_1 a_2}}{3} \left(1 + \frac{h}{\lambda} \right)^{3/2} - 2(\lambda + h) - \log \left(a_1 a_2 \left(1 + \frac{h}{\lambda} \right)^2 \right) \\ &+ \frac{\lambda}{2} + \log \lambda - 0.15. \end{aligned}$$

Now we consider the three cases for i .

$$\boxed{i = 1}$$

We have seen above that in this case $a < 0, b < 0$ and $a \approx b$, for this reason we put $c = a - b$ and rewrite the linear form Λ as

$$\Lambda_1 = \Lambda = \log |\delta_1| - c \log \left| \frac{\varepsilon_3}{\varepsilon_2} \right| + a \log \left| \frac{\xi_3 \varepsilon_3}{\xi_2 \varepsilon_2} \right| = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

with

$$b_1 = |a|, \quad \alpha_1 = \left| \frac{\xi_3 \varepsilon_3}{\xi_2 \varepsilon_2} \right|, \quad b_2 = 1, \quad \alpha_2 = |\delta_1 (\varepsilon_2 / \varepsilon_3)^c|^\sigma \quad \text{where } \sigma \in \{-1, +1\}.$$

Put $\ell_1 = \log \alpha_1$, $\ell'_1 = \log \alpha_2$ then

$$\Lambda = |a|\ell_1 - \ell'_1.$$

1. *Estimating ℓ_1*

One can verify that the minimal polynomial for $\varepsilon\xi = \xi + \xi^2$ is

$$G(X) = X^3 - (n^2 + 3n + 2)X^2 - (2n + 3)X - 1$$

and that

$$\begin{aligned} G\left(-\frac{1}{n} + \frac{1}{n^2}\right) &> 0, \\ G\left(-\frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3}\right) &< 0, \\ G\left(-\frac{1}{n} + \frac{2}{n^2} - \frac{4}{n^3}\right) &< 0, \\ G\left(-\frac{1}{n} + \frac{2}{n^2} - \frac{5}{n^3}\right) &> 0. \end{aligned}$$

Since the function $x \mapsto x(1-x)$ is increasing for $0 < x < 0.5$, we have

$$\left(1 - \frac{1}{n}\right) \frac{1}{n} < |\xi_3 \varepsilon_3| < \left(1 - \frac{1}{n-1}\right) \frac{1}{n-1}.$$

For similar reasons,

$$\left(1 - \frac{1}{n+1}\right) \frac{1}{n+1} < |\xi_2 \varepsilon_2| < \left(1 - \frac{1}{n}\right) \frac{1}{n}.$$

Which implies $|\xi_3 \varepsilon_3|/|\xi_2 \varepsilon_2| > 1$.

These remarks show that

$$\begin{aligned} \frac{1}{n} - \frac{1}{n^2} &< |\xi_3 \varepsilon_3| < \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} \\ \frac{1}{n} - \frac{2}{n^2} + \frac{4}{n^3} &< |\xi_2 \varepsilon_2| < \frac{1}{n} - \frac{2}{n^2} + \frac{5}{n^3} \end{aligned}$$

thus

$$1 + \frac{1}{n+5} \leq \frac{n^2 - n}{n^2 - 2n + 5} < \left| \frac{\xi_3 \varepsilon_3}{\xi_2 \varepsilon_2} \right| < \frac{n^2 - n + 1}{n^2 - 2n + 4} < 1 + \frac{1}{n}$$

and

$$\frac{1}{n+6} < \ell_1 < \frac{1}{n}.$$

2. Estimating ℓ'_1

We have

$$1 + \frac{1 - \frac{2n}{n^2-1}}{n+2} < \delta_1 = 1 + \frac{\xi_3 - \xi_2}{\xi_1 - \xi_3} < 1 + \frac{1}{n+1},$$

thus $0 < \log \delta_1 < \frac{1}{n+1}$, and moreover

$$n - 1.5 < n - 1 - \frac{1}{n-1} = \frac{1 - \frac{1}{n-1}}{1/n} < \left| \frac{\varepsilon_3}{\varepsilon_2} \right| < \frac{1 - \frac{1}{n}}{1/(n+1)} = n - \frac{1}{n}.$$

So that

$$|c| \log(n - 1.5) - \frac{1}{n+1} < \ell'_1 < |c| \log n + \frac{1}{n+1}.$$

As a consequence of the estimates of ℓ_1 , ℓ'_1 and $|\Lambda|$, we have

$$n(|c| \log(n - 1.5) - 1/n) \leq |a| \leq (n+6)(|c| \log n + 1/n).$$

3. Estimating measures

We have

$$M(\xi_3 \varepsilon_3 / (\xi_2 \varepsilon_2)) = |\xi_1 \varepsilon_1|^4 |\xi_2 \varepsilon_2|^2 < (n+2)^4 (n+3)^4 n^{-2} < (n+4)^6$$

and

$$M(\xi_3 / \xi_2) \leq (n+2)^4, \quad M(\delta_1) \leq (n+2)^6.$$

4. Application of Proposition 1

We have to take

$$h \geq \max \left\{ 5\lambda, D \log \left(\frac{|a|}{a_2} + \frac{1}{a_1} \right) + \log \lambda + 1.56 \right\},$$

by the upper bound of $|a|$ we choose

$$h = \max \left\{ 5\lambda, 6 \log \left(\frac{(n+6)(|c| \log n + 1/n)}{a_2} + \frac{1}{a_1} \right) + \log \lambda + 1.56 \right\},$$

and we can choose

$$a_1 = \max \{ 2\lambda, (\rho - 1)/n + 12 \log(n+4) \},$$

$$a_2 = \max \left\{ 2\lambda, (\rho - 1)(|c| \log n + 1/n) + 12 \left(1 + \frac{2|c|}{3} \right) \log(n+4) \right\}.$$

Applying inequality (iii), we get,

$$\log |\Lambda_1| \geq -L_1, \quad (\text{say}).$$

5. Upper bound on n

In this case, using (8)₁ and (7), we get

$$\begin{aligned} \log |\Lambda_1| &\leq -3A \frac{R}{\log(n+3)} + \frac{3}{2} \log(n+3) + 2 \\ &\leq -3A \frac{\log^2(n-1)}{\log(n+3)} + \frac{3}{2} \log(n+3) + 2, \end{aligned}$$

where $A \geq |a|$.

We have already seen that

$$|a| \geq n(|c| \log(n-1.5) - 1/n).$$

When $c \neq 0$, choosing $\rho = 67.1$ and combining the previous inequalities, we get

$$n \leq 150000.$$

6. The case $c = 0$

In the special case $c = 0$, we have

$$\Lambda_1 = \log |\delta_1| + a \log \left| \frac{\xi_3 \varepsilon_3}{\xi_2 \varepsilon_2} \right|.$$

By the estimates of δ_1 , $\frac{\xi_3 \varepsilon_3}{\xi_2 \varepsilon_2}$ and $|\Lambda|$ we have

$$|a| \leq \frac{1/n}{1/(n+6)} = \frac{n+6}{n} < 2.$$

The case $a = b = 1$ gives $x - \xi y = \xi + \xi^2$ which is impossible. Whereas the case $a = b = -1$ gives

$$\begin{aligned} |y| &= \left| \frac{(\xi_1 \varepsilon_1)^{-1} - (\xi_2 \varepsilon_2)^{-1}}{\xi_2 - \xi_1} \right| < \frac{\left(\left(1 - \frac{1}{n}\right) \frac{1}{n+1} \right) + (n+1)^{-2}}{n+1} \\ &< \frac{\frac{n(n+1)}{n-1} + \frac{1}{n-1}}{n+1} = \frac{n+1}{n-1}, \end{aligned}$$

so that $|y| \leq 1$, and this has been studied above.

$$\boxed{i = 2}$$

Here we choose $j = 3$, $k = 1$ and put $b = -2a + c - 1$ (recall that $a < 0$ and $b > 0$) and rewrite Λ as

$$\Lambda_2 = \Lambda = \log \delta'_2 + c \log \left| \frac{\varepsilon_1}{\varepsilon_3} \right| + a \log \left| \frac{\xi_1 \varepsilon_3^2}{\xi_3 \varepsilon_1^2} \right| = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

with

$$b_1 = |a|, \quad \alpha_1 = \left| \frac{\xi_3 \varepsilon_1^2}{\xi_1 \varepsilon_3^2} \right|, \quad b_2 = 1, \quad \alpha_2 = |\delta'_2 (\varepsilon_1 / \varepsilon_3)^c|^{\sigma} \quad \text{where } \sigma \in \{-1, +1\},$$

and where

$$\delta'_2 = \frac{\varepsilon_3 (\xi_1 - \xi_2)}{\varepsilon_1 (\xi_3 - \xi_2)}.$$

Now we put $\ell_2 = \log \alpha_1$, $\ell'_2 = \log \alpha_2$ then

$$\Lambda_2 = |a| \ell_2 - \ell'_2.$$

1. *Estimating ℓ_2*

Using the estimate R3 we get

$$\begin{aligned} & \frac{(n+1) \left(1 - \frac{1}{n-1/(n^2-n-1)}\right)^2}{\frac{1}{n-1/(n^2-n)}(n+2+n^{-2})^2} \\ & \leq \alpha_1 = \left| \frac{\xi_1 \varepsilon_3^2}{\xi_3 \varepsilon_1^2} \right| \leq \frac{(n+1+n^{-2}) \left(1 - \frac{1}{n-1/(n^2-n)}\right)^2}{\frac{1}{n-1/(n^2-n-1)}(n+2)^2}, \end{aligned}$$

from which we can deduce

$$1 + \frac{5}{n} < \alpha_1 < 1 + \frac{5}{n} + \frac{11}{n^2}.$$

Thus

$$\frac{5}{n+3} < \frac{5}{n} - \frac{13}{n^2} < \ell_2 < \frac{5}{n} + \frac{11}{n^2} < \frac{5}{n-3}.$$

2. *Estimating ℓ'_2*

Here,

$$\delta'_2 = \frac{\varepsilon_3(\xi_1 - \xi_2)}{\varepsilon_1(\xi_3 - \xi_2)} = 1 + \frac{\varepsilon_2(\xi_1 - \xi_3)}{\varepsilon_1(\xi_3 - \xi_2)},$$

which implies

$$\begin{aligned} & 1 + \frac{1}{n} < 1 + \frac{1}{(n+2) \left(1 - \frac{2}{n+1}\right)} < \delta'_2 < 1 + \frac{\frac{n+2}{n}}{(n+1) \left(1 - \frac{2n}{n^2-1}\right)} \\ & = 1 + \frac{1+2/n}{n-1-2/(n-1)} < 1 + \frac{n-1}{(n-1)^2-2} + \frac{2}{(n-1)^2-2} = 1 + \frac{n+1}{n^2-2n-1}, \end{aligned}$$

so that $1 + \frac{1}{n} < \delta'_2 < 1 + \frac{1}{n-3}$ for $n \geq 4$. Also

$$n+3 < \frac{n+2}{1 - \frac{1}{n-1}} < \left| \frac{\varepsilon_1}{\varepsilon_3} \right| < \frac{n+2+1/n^2}{1 - \frac{1}{n}} < n+4.$$

So that

$$|c| \log(n+3) - \frac{1}{n-3} < \ell'_2 < |c| \log(n+4) + \frac{1}{n-3}.$$

As a consequence of the estimates of ℓ_2 , ℓ'_2 and $|\Lambda|$, we have

$$\frac{n-3}{5}(|c|\log(n+3) - 1/(n-4)) < |a| < \frac{n+3}{5}(|c|\log(n+4) + 1/(n-4)).$$

3. Estimating measures

Here

$$\delta'_2 = \frac{\varepsilon_1(\xi_2 - \xi_3)}{\varepsilon_3(\xi_2 - \xi_1)},$$

and

$$h(\delta'_2) \leq \frac{5}{3} \log(n+2), \quad h\left(\frac{\varepsilon_1}{\varepsilon_3}\right) \leq \frac{2}{3} \log(n+2).$$

Moreover

$$M(\xi_1 \varepsilon_3^2 / (\xi_3 \varepsilon_1^2)) \leq (n+2)^6.$$

[Look at the conjugates of modulus > 1 of this number.]

4. Application of Proposition 1

We take

$$h = \max \left\{ 5\lambda, 6 \log \left(\frac{(n+3)(|c|\log(n+4) + 1)}{5a_2} + \frac{1}{a_1} \right) + \log \lambda + 1.56 \right\}$$

and we can choose

$$a_1 = \max \left\{ 2\lambda, \frac{5}{n-3}(\rho-1) + 12 \log(n+4) \right\},$$

$$a_2 = \max \left\{ 2\lambda, (\rho-1)(|c|\log(n+4) + 1/(n-4)) + 12 \left(\frac{5}{3} + \frac{2}{3}|c| \right) \log(n+4) \right\}.$$

By Proposition 1,

$$\log |\Lambda_2| \geq -L_2, \quad (\text{say}).$$

5. Upper bound on n

By (8),

$$\log |\Lambda_2| \leq -\frac{3A}{2} \frac{R}{\log(n+3)} + \frac{3}{2} \log(n+3) + 2.$$

We have seen that

$$|a| \geq \frac{n-3}{5} (|c| \log(n+3) - 1/(n-4)),$$

this implies

$$A = |b| \geq |a| \left(2 - \frac{5}{(n-3) \log n} \right) - 2.$$

When $c \neq 0$, choosing $\rho = 81.2$ we get

$$n \leq 810000.$$

6. The special case $c = 0$

If $c = 0$ then the relations $\Lambda_2 = |a|\ell_2 - \log \delta'_2$, $\ell_2 > 5/(n-3)$ and $\log(1+1/n) < \log \delta'_2 < 1/(n-3)$ imply

$$|\Lambda_2| \geq \min \left\{ \frac{5}{n+3} - \frac{1}{n-3}, \frac{1}{n} - \frac{1}{2n^2} \right\}$$

in contradiction with (9).

$i = 3$

Here $j = 1$ and $k = 2$, put $a = -2b + c + 1$ and rewrite Λ as

$$\Lambda_3 = \Lambda = \log \delta'_3 + c \log \left| \frac{\xi_2}{\xi_1} \right| + b \log \left| \frac{\xi_1^2 \varepsilon_2}{\xi_2^2 \varepsilon_1} \right| = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

with

$$b_1 = |b|, \quad \alpha_1 = \left| \frac{\xi_1^2 \varepsilon_2}{\xi_2^2 \varepsilon_1} \right|, \quad b_2 = 1, \quad \alpha_2 = \left| \delta'_3 (\xi_2 / \xi_1)^c \right|^\sigma \quad \text{where } \sigma \in \{-1, +1\},$$

and where

$$\delta'_3 = \frac{\xi_2(\xi_1 - \xi_3)}{\xi_1(\xi_2 - \xi_3)}.$$

Put $\ell_3 = \log \alpha_1$, $\ell'_3 = \log \alpha_2$ then

$$\Lambda_3 = |b|\ell_3 - \ell'_3.$$

1. *Estimating ℓ_3*

One can prove that

$$\frac{1}{n} < \log \left(1 + \frac{1}{n} + \frac{3}{n^2} \right) < \ell_3 < \frac{1}{n} + \frac{4}{n^2}.$$

2. *Estimating ℓ'_3*

One can also prove that

$$|c| \log(n+1) - \frac{1}{n} - \frac{5}{n^2} < \ell'_3 < |c| \log(n+4) + \frac{1}{n} + \frac{5}{n^2}.$$

As a consequence of the estimates of ℓ_3 , ℓ'_3 and $|\Lambda|$, we have

$$\frac{n}{1+4/n} \left(|c| \log(n+1) - \frac{1}{n} - \frac{5}{n^2} \right) < |b| < n \left(|c| \log(n+4) + \frac{1}{n} + \frac{5}{n^2} \right).$$

3. *Estimating measures*

One has

$$h(\delta_3) \leq \log(n+2), \quad h(\xi_2/\xi_1) \leq \frac{2}{3} \log(n+2), \quad h\left(\frac{\xi_1^2 \varepsilon_2}{\xi_2^2 \varepsilon_1}\right) \leq \log n.$$

4. *Application of Proposition 1*

We take

$$h = \max \left\{ 5\lambda, 6 \log \left(\frac{n(|c| \log(n+1) + 1)}{a_2} + \frac{1}{a_1} \right) + \log \lambda + 1.56 \right\}$$

and we can choose

$$a_1 = \max \left\{ 2\lambda, \frac{1}{n}(\rho - 1) + 12 \log(n+4) \right\},$$

$$a_2 = \max \left\{ 2\lambda, \left(|c| + \frac{1}{n} \right) (\rho - 1) \log(n+4) + 12 \left(1 + \frac{2}{3} |c| \right) \log(n+4) \right\}.$$

By Proposition 1,

$$\log |\Lambda_3| \geq -L_3, \quad (\text{say}).$$

5. Upper bound on n

We have

$$\log |\Lambda_3| \leq \frac{3A}{2} \frac{R}{\log(n+3)} + \frac{3}{2} \log(n+3) + 2.$$

We have seen that

$$|b| \geq \frac{n}{1+4/n} \left(|c| \log(n+1) - \frac{1}{n} - \frac{5}{n^2} \right),$$

this implies

$$A = |a| \geq |b| \left(2 - \frac{1+4/n}{n \log n} \right) - 2.$$

When $c \neq 0$, choosing $\rho = 48.3$ we get

$$n \leq 260000.$$

6. The special case $c = 0$

If $c = 0$ then $b = -1$ and $a = 3$, and

$$|y| = \left| \frac{\xi_1^3 \varepsilon_1^{-1} - \xi_2^3 \varepsilon_2^{-1}}{\xi_2 - \xi_1} \right| < \frac{(n+1+1/n)^2}{n+2-1/n} < n+2,$$

in contradiction with the hypothesis $|y| \geq (n^2 - 3)/2$.

Application of a theorem of M. Waldschmidt

Let α_i , $1 \leq i \leq n$ be non-zero algebraic numbers and $b_1, b_2, \dots, b_n$ be positive rational integers and suppose that the number

$$\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_n$$

is not zero. We apply a theorem of M. WALDSCHMIDT [W], Corollaire 1.5.

Put $D = [Q(\alpha_1, \dots, \alpha_n) : Q]$ and $g = [R(\log \alpha_1, \dots, \log \alpha_n) : R]$. For $1 \leq i \leq n$, let $A_i > 1$ be real numbers such that $\log A_i \geq h(\alpha_i)$. Then the quoted result is the following:

Proposition 2. *Let E and f be positive real numbers, $E \geq e$ such that,*

$$E \leq \min \left\{ A_1^D, \dots, A_n^D, \frac{nD}{f} \left(\sum_{i=1}^n \frac{|\log \alpha_i|}{|\log A_i|} \right)^{-1} \right\}.$$

Put

$$Z_0 = \max \left\{ 7 + 3 \log n, \frac{g}{D} \log E, \log \left(\frac{D}{\log E} \right) \right\},$$

$$M = \max_{1 \leq j < n} \left\{ \frac{b_n}{\log A_j} + \frac{b_j}{\log A_n} \right\},$$

$$G_0 = \max\{4nZ_0, \log M\},$$

$$U_0 = \max\{D^2 \log A_1, \dots, D^2 \log A_n, D^{n+2} G_0 Z_0 \log A_1 \cdots \cdots \log A_n (\log E)^{-n-1}\}.$$

Then

$$|\Lambda| \geq \exp\{-1500 g^{-n-2} 2^{2n} n^{3n+5} (1 + g/f)^n U_0\}.$$

In the present case we have three logarithms, $D = 6$, $g = 1$ and, for $n \geq 3$ (here n is again the parameter of our cubic equations), we can take

$$\log A_1 = \log(n+2), \quad \log A_2 = \log A_3 = \frac{2}{3} \log(n+2),$$

and

$$E = e, \quad f = 3/e, \quad Z_0 = 7 + 3 \log 3, \quad G_0 = \max\{12Z_0, \log M\}.$$

A short computation shows that Proposition 2 implies

$$\log |\Lambda| > \begin{cases} -1.398 \times 10^{19} \times \log^3(n+2), & \text{if } \log M < 123.6, \\ -1.132 \times 10^{17} \times \log M \times \log^3(n+2), & \text{otherwise.} \end{cases}$$

We can take

$$M = \frac{3}{2 \log(n+2)} + \frac{A}{\log(n+2)} < \frac{A+2}{\log(n+2)}.$$

Using the upper bound (10) on $\log |\Lambda|$ proved before (we get

$$\frac{3A}{2} \frac{R}{\log(n+3)} \leq C \max \left\{ 123.6, \log \left(\frac{A+2}{\log(n+2)} \right) \right\} \times \log^3(n+2) \\ + 2 \log(n+3) + 3,$$

where $C = 1.398 \times 10^{19}$. Which gives the following upper bound for A in terms of n :

$$A \leq \left(\frac{2C}{3} \max \left\{ 123.6, \log \left(\frac{A+2}{\log(n+2)} \right) \right\} \times \log^3(n+2) + 2 \log(n+3) + 3 \right) \\ \times \frac{\log(n+3)}{R}.$$

Using the upper bound on n , we find $A < 1.1 \times 10^{23}$.

4. Application of Diophantine approximation

We use the following lemma which is a variant of a result of Baker–Davenport.

Lemma. *Let $\Lambda = u\alpha + v\beta + \gamma$, where α , β and γ are nonzero real numbers and where u and v are rational integers, with $|u| \leq A$. Let $Q > 0$ be a real number. Suppose that θ_1 and θ_2 satisfy*

$$|\theta_1 - \alpha/\beta| < \frac{1}{100Q^2}, \quad \text{and} \quad |\theta_2 - \gamma/\beta| < \frac{1}{Q^2}.$$

Let p/q be a rational number with $1 \leq q \leq Q$ and $|\theta_1 - p/q| < 1/q^2$ and suppose that $q\|q\theta_2\| \geq 1.01, A+2$, [where $\|\cdot\|$ denotes the distance to the nearest integer] then

$$|\Lambda| \geq \frac{|\beta|}{Q^2}.$$

PROOF. Put $|\Lambda| = \eta$, then

$$\left| q \frac{\Lambda}{\beta} \right| = \left| uq \left(\frac{\alpha}{\beta} - \theta_1 \right) + u(q\theta_1 - p) + pu + vq + q \left(\frac{\gamma}{\beta} - \theta_2 \right) + q\theta_2 \right| = \frac{q\eta}{|\beta|}.$$

Hence,

$$\begin{aligned} q\|q\theta_2\| &\leq q \left(\frac{q\eta}{|\beta|} + |uq| \left| \frac{\alpha}{\beta} - \theta_1 \right| + \frac{|u|}{q} + q \left| \frac{\gamma}{\beta} - \theta_2 \right| \right) \\ &< \frac{q^2\eta}{|\beta|} + \frac{|u|q^2}{100Q^2} + |u| + \frac{q^2}{Q^2} \leq \frac{Q^2\eta}{|\beta|} + 1.01A + 1, \end{aligned}$$

which leads at once to the result. $\square$

We applied the above lemma for $n \leq 150000$, $n \leq 810000$ and $n \leq 260000$ respectively in the three cases $i = 1, 2$ and 3 . We found no non-trivial solution for $n \geq 10$. The verification took less than six hours on a DEC *alpha* Station 1000A.

Thus, we have proved the Theorem stated in the Introduction.

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