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Additive functions with monotonic norm

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To Kálmán Győry and András Sárközy,
on the occasion of their 60th birthday

Abstract. We describe all Hilbert-space valued additive functions with monotonic norm.

1. Introduction

A classical 1946 theorem of Erdős asserts that if a real-valued additive arithmetical function is monotonic, it must be of the form $c \log n$ with some constant c. Since then many variants and generalizations have been published. Among others, Katalin Kovács [1] proved that if f is an additive function with values in $\mathbb{R}^d$ and $|f(n)|$ is increasing from some point on, then f must be of the form $a \log n$ with some constant $a \in \mathbb{R}^d$. She also pointed out that the theorem cannot be extended to functions with values in an infinite dimensional Hilbert space. Indeed, if H is an infinite dimensional Hilbert space and $e_p \in H$ are orthogonal unit vectors for each prime p, then the additive function defined at prime powers by $f(p^k) = \sqrt{\log p^k} e_p$ will satisfy $|f(n)| = \sqrt{\log n}$ for all n.

In this paper we describe all Hilbert-space valued additive functions with monotonic norm.

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While a Hilbert-space valued arithmetical function may not be the most natural object, this is the first instance that the class of "regular" functions in some sense can be completely described and it is genuinely wider than the set of logarithmic functions.

Theorem 1.1. *Let H be a Hilbert space, $f : \mathbb{N} \to H$ an additive function and suppose that $|f(n)|$ is monotonic for $n > n_0$. Then there are nonnegative real numbers α, β such that*

$$|f(n)| = \alpha(\log n)^2 + \beta \log n \tag{1.1}$$

for all n. If $\beta = 0$, then $f(n) = a \log n$ for all n with some $a \in H$ satisfying $|a| = \sqrt{\alpha}$. If $\beta > 0$, then H is infinite dimensional, and there are nonempty orthogonal subspaces E_p for each prime p, and a vector a satisfying $|a| = \sqrt{\alpha}$ and orthogonal to all the E_p such that f is of the form $f(n) = a \log n + g(n)$, and the additive function g satisfies

$$g(p^k) \in E_p, \quad |g(p^k)| = \sqrt{\beta \log p^k} \tag{1.2}$$

for every prime-power p^k. Conversely, every function defined in this way satisfies (1.1)*, thus its norm is monotonic.*

2. Quadritive functions

We will study the functions that can occur as $|f(n)|^2$. By considering the coordinates we see that

$$|f(n)|^2 = \sum f_i(n)^2$$

with additive functions f_i. We will consider a slightly wider class which can be characterized by a functional equation.

Definition 2.1. We call a real-valued arithmetical function F *quadritive* (from quadratic + additive) if it satisfies

$$F(abc) + F(a) + F(b) + F(c) = F(ab) + F(bc) + F(ca) \tag{2.1}$$

for all pairwise coprime triplets of natural numbers a, b, c.

In this section we establish some basic properties of quadritive functions.

Statement 2.2. *For a real arithmetical function the following are equivalent.*

a) *There are additive functions f_i and constants $c_i = \pm 1$ such that*

$$F(n) = \sum_{i=1}^{\infty} c_i f_i(n)^2 \tag{2.2}$$

for all n.

b) *F can be expressed in the form*

$$F(n) = \sum_{i=1}^{\infty} f_i(n) g_i(n) \tag{2.3}$$

with additive functions f_i, g_i.

c) *F is quadritive.*

In a) *and* b), *the series are assumed to be pointwise convergent. Furthermore, every quadritive function has representations in forms* (2.2) *and* (2.3) *with the following stronger convergence property: for each n the series have only finitely many nonzero terms.*

Lemma 2.3. *If a quadritive function vanishes on the set of numbers of the form p^k and $p^k q^l$, where p, q are primes and k, l are natural numbers, then it is identically 0.*

Proof. Suppose that our function is not identically 0; then there is a minimal n with $F(n) \neq 0$. By substituting $a = b = c = 1$ into (2.1) we see that $F(1) = 0$, so this number satisfies $n > 1$. It cannot be a prime-power or a product of two prime-powers by assumption, thus its canonical decomposition contains at least 3 different primes. Let a, b be two prime-powers from the canonical decomposition of n and put $c = n/ab$. Consider equation (2.1) with these numbers a, b, c. We have $a, b, c, ab, bc, ca < n$, thus all terms vanish save $F(abc) = F(n)$, a contradiction. □

Lemma 2.4. *Every real function F_0, defined on the set of natural numbers of the form p^k or $p^k q^l$, where p, q are primes and k, l are natural numbers, can be extended to a function of the form* (2.3). *Furthermore, this can be done so that in* (2.3) *for every fixed n there are only finitely many nonzero values of $f_i(n)$ and $g_i(n)$.*

Proof. Define, for each prime-power p^k, an additive function h_{p^k} by

$$h_{p^k}(n) = \begin{cases} 1, & \text{if } p^k \| n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we put

$$F = \sum_{p,k} F_0(p^k) h_{p^k}^2 + \sum_{p,q,k,l} \big(F_0(p^k q^l) - F_0(p^k) - F_0(q^l)\big) h_{p^k} h_{q^l}.$$

After a suitable rearrangement, this becomes an expression of type (2.3). If evaluated at any natural number n, there are only finitely many nonzero terms, thus the series will be automatically convergent. It is immediate to check that it extends F_0. □

Proof of Statement 2.2. To see that property a) implies quadritivity, it is sufficient to check (2.1) for functions of the form $F(n) = f(n)^2$ with additive f. For such a function we have

$$\begin{aligned} F(abc) &= \big(f(abc)\big)^2 = \big(f(a) + f(b) + f(c)\big)^2 \\ &= f(a)^2 + f(b)^2 + f(c)^2 + 2\big(f(ab) + f(bc) + f(ca)\big). \end{aligned}$$

Expanding $F(ab)$, $F(bc)$, $F(ca)$ similarly we immediately see that (2.1) holds.

Next we prove that quadritivity implies properties b) and a). Let F be a quadritive function. Let G be a function with property b) that coincides with F on prime-powers and products of two prime-powers; such a function exists by Lemma 2.4. This function also has a representation in form (2.2). To see this we apply the identity

$$fg = \left(\frac{f+g}{2}\right)^2 - \left(\frac{f-g}{2}\right)^2$$

to each summand of (2.3). Since the representation provided by Lemma 2.4 had the property that for each n there were only finitely many nonzero values of $f_i(n)$ and $g_i(n)$, the resulting series (2.2) will also be finite for every n. The function G is quadritive by the already proved implication b) $\longrightarrow$ c).

Finally to deduce condition b) from a) we simply put $g_i = c_i f_i$. □

Statement 2.5. *Every real function F_0, defined on the set of natural numbers of the form p^k or $p^k q^l$, where p, q are primes and k, l are natural numbers, can be extended to a quadritive function, and this extension is unique.*

Proof. We take the extension provided by Lemma 2.4; it is quadritive by Statement 2.2, and unicity follows from Lemma 2.3. □

Definition 2.6. We call a real-valued arithmetical function F *completely quadritive*, if it satisfies (2.1) for all, not necessarily coprime triplets of natural numbers a, b, c.

Statement 2.7. *For a real arithmetical function the following are equivalent.*

a) *There are completely additive functions f_i, f^* and constants $c_i = \pm 1$ such that*

$$F(n) = f^*(n) + \sum_{i=1}^{\infty} c_i f_i(n)^2 \tag{2.4}$$

for all n.

b) *F can be expressed in the form*

$$F(n) = f^*(n) + \sum_{i=1}^{\infty} f_i(n) g_i(n) \tag{2.5}$$

with completely additive functions f_i, g_i and f^*.

c) *F is completely quadritive.*

We will not need this result, so we only outline the proof. It is completely analogous to that of Statement 2.2 with one exception. An additive function is also quadritive, and similarly a completely additive function is completely quadritive. From the proof of Statement 2.2 we learned as a byproduct that an additive function can be expressed as a sum of squares of additive function; however, the corresponding statement for completely additive functions fails. (Indeed, a sum of squares of completely additive function always satisfies $F(p^2) = 4F(p)$.) Thus the only difference is in the formulation and proof of the analog of Lemma 2.4, which proceeds as follows.

Lemma 2.8. *Every real function F_0, defined on the set of natural numbers of the form p or pq, where p, q are primes, can be extended to a function of the form* (2.5).

Proof. Define, for each prime p, a completely additive function h_p by

$$h_p(n) = k, \quad \text{if } p^k \| n.$$

Then we put

$$F = \sum_p \left\{ \left(2F_0(p) - \frac{1}{2}F_0(p^2) \right) h_p - \left(F_0(p) - \frac{1}{2}F_0(p^2) \right) h_p^2 \right\} + \sum_{p<q} \big(F_0(pq) - F_0(p) - F_0(q) \big) h_p h_q.$$

If evaluated at any natural number n, there are only finitely many nonzero terms, thus the series will be automatically convergent and so it defines a completely quadritive function. It is immediate to check that it extends F_0.

After combining the linear terms into a single completely additive function

$$f^* = \sum_p \left(2F_0(p) - \frac{1}{2}F_0(p^2) \right) h_p$$

and a suitable rearrangement, this becomes an expression of type (2.5). □

Statement 2.9. *Every real function F_0, defined on the set of natural numbers of the form p or pq, where p, q are (not necessarily distinct) primes, can be extended to a completely quadritive function, and this extension is unique.*

The proof of this statement is completely analogous to the proof of Statement 2.5.

3. Regular quadritive functions

In this section we describe the monotonic quadritive functions.

Theorem 3.1. *Let F be a quadritive function. If there is an n_0 such that $F(n)$ is monotonic for $n > n_0$, then F is of the form*

$$F(n) = \alpha(\log n)^2 + \beta \log n \tag{3.1}$$

with constants α, β.

The proof will proceed through several lemmas.

Lemma 3.2. *If a quadritive function F is monotonic for $n > n_0$, then*

$$F(n) = O\big((\log n)^3\big). \tag{3.2}$$

Proof. Assume that F is increasing for $n > n_0$. If it is always negative, it is bounded and we are done. Assume it is eventually positive.

We shall find a number $n_1 \geq n_0$, take a positive number a such that

$$F(n) \leq a(\log n)^3 \tag{3.3}$$

holds for all $n \leq n_1$, and show by induction that (3.3) holds for all n. We describe the inductive step, and during this we will find conditions for n_1 that will guarantee that the induction works.

Assume we know (3.3) for all numbers $< n$. We try to deduce it for n as follows. Let m be the smallest even integer such that

$$n \leq (m-1)m(m+1). \tag{3.4}$$

The numbers $m-1, m, m+1$ are pairwise coprime, thus we have

$$\begin{aligned} F(n) &\leq F\big((m-1)m(m+1)\big) \\ &= F\big((m-1)m\big) + F\big((m-1)(m+1)\big) + F\big(m(m+1)\big) \\ &\quad - F(m-1) - F(m) - F(m+1). \end{aligned}$$

(3.4) shows $m > n^{1/3}$, thus if n_1 is so large that $f(n) > 0$ for $n > n_1^{1/3}$, then we can conclude

$$F(n) \leq F\big((m-1)m\big) + F\big((m-1)(m+1)\big) + F\big(m(m+1)\big) \leq 3F\big(m(m+1)\big),$$

where the last step follows from monotonicity.

This m satisfies $m \sim n^{1/3}$, thus $m(m+1) \sim n^{2/3}$. If n is large enough, then $m(m+1) \leq 2n^{2/3} < n^{3^{-1/3}}$; this follows from $n > n_1$ if n_1 exceeds a certain absolute constant. Then we can use the induction hypothesis to obtain

$$F(n) \leq 3a\left(\log n^{3^{-1/3}}\right)^3 = a\big(\log n\big)^3$$

as wanted. □

Lemma 3.4. *If a quadritive function F is monotonic for $n > n_0$, then*

$$F(n+1) - F(n) \to 0$$

for almost all n (in the sense of asymptotic density).

Proof. Assume that F is increasing. By the previous lemma we have

$$-b \le F(n) \le a(\log n)^3$$

for all n with certain positive constants a, b. Then $|F(n+1) - F(n)| > \varepsilon$ can have at most

$$\frac{1}{\varepsilon}\big(b + a(\log x)^3\big) = o(x)$$

solutions in numbers $n_0 \le n \le x$. □

Lemma 3.4. *If a quadritive function F satisfies*

$$F(n+1) - F(n) \to 0 \text{ for almost all } n, \tag{3.5}$$

then equation (2.1) *holds for all triplets satisfying $(a,b) = (a,c) = 1$.*

Proof. By a repeated application of (3.5) we see that $F(n+k) - F(n) \to 0$ for any fixed k for a sequence of n's of density 1.

We will select integers x, y coprime to abc and apply equation (2.3) to the following triplets: (a, b, x), (a, c, y), (a, bc, x). By adding the first two equations and substracting the third we obtain, after some cancellation and rearrangement,

$$\begin{aligned} F(abc) + F(a) + F(b) + F(c) - F(ab) - F(bc) - F(ca) \\ = \big(F(bx) - F(y)\big) + \big(F(abcx) - F(acy)\big) \\ - \big(F(abx) - F(ay)\big) - \big(F(bcx) - F(cy)\big). \end{aligned} \tag{3.6}$$

We put $x = abct + 1$, $y = ab^2ct + 1$. This choice guarantees the required coprimality assumptions and also $bx - y = b - 1$ independently of t. Consequently the first term of the right side of (3.6) tends to 0 for almost all t, and so do the other terms by a similar argument. So the right side of (3.6) tends to 0 for almost all t, consequently the left side, which is independent of t, must be 0. □

Lemma 3.5. *If a quadritive function F satisfies $F(n+1)-F(n)\to 0$ for almost all n, then equation* (2.1) *holds for all triplets (that is, F is completely quadritive).*

PROOF. We select integers x, y coprime to abc and apply equation (2.3) to the same triplets (a,b,x), (a,c,y), (a,bc,x). We can do this, since these triplets satisfy the condition of the previous lemma. We obtain (3.6) again. From now on we can literally repeat the proof of the previous lemma. □

Lemma 3.6. *Any quadritive function which is monotonic from some point on is completely quadritive.*

PROOF. Follows from Lemmas 3.3 and 3.5. □

Lemma 3.7. *Let F be completely quadritive and $n>1$ an integer. There are real numbers α_n, β_n such that*

$$F(n^k)=\alpha_n(\log n^k)^2+\beta_n\log n^k \tag{3.7}$$

for every nonnegative integer k.

PROOF. On substituting $a=b=n$, $c=n^k$ into (2.1) we obtain

$$F(n^{k+2})=2F(n^{k+1})-F(n^k)+F(n^2)-2F(n). \tag{3.8}$$

Now select α, β so that (3.7) holds for $k=1,2$. Using (3.6) an easy induction shows that it will hold for all $k\ge 3$. It holds for $k=0$ automatically, since both sides of (3.7) vanish. □

PROOF of Theorem 3.1. Take a quadritive function F monotonically increasing from a point on. By Lemma 3.6 we know that F is completely quadritive, thus by the previous lemma for each n there are constants α_n, β_n such that

$$F(n^k)=\alpha_n(\log n^k)^2+\beta_n\log n^k.$$

Since this is an increasing function of k, we know that $\alpha_n\ge 0$. We will show that α_n and β_n are independent of n. To this end take another integer $m\ne n$. Take two large integers x, y satisfying

$$x=\left[y\frac{\log n}{\log m}\right],$$

so that $m^x \le n^y$ "but just". We have

$$\alpha_m(\log m^x)^2 + \beta_m \log m^x \le \alpha_n(\log n^y)^2 + \beta_n \log n^y. \tag{3.9}$$

Since $\log m^x = \log n^y + O(1)$, the left side is $= \alpha_m(\log n^y)^2 + O(\log n^y)$, and this shows that $\alpha_m \le \alpha_n$. By exchanging the roles of m and n we see that $\alpha_n \le \alpha_m$, so that $\alpha_m = \alpha_n = \alpha$, say.

Now (3.9) reduces to

$$\beta_m \log m^x - \beta_n \log n^y \le \alpha(\log n^y - \log m^x)(\log n^y + \log m^x). \tag{3.10}$$

Introduce the notation

$$\delta = \log n^y - \log m^x = \log m \left\{ y \frac{\log n}{\log m} \right\}.$$

The left side of (3.10) can be rewritten as $(\beta_m - \beta_n)\log n^y - \beta_m\delta$, and the right side as $\alpha\delta(\log n^y + \log m^x) \le 2\alpha\delta \log n^y$. Thus we obtain

$$(\beta_m - \beta_n)\log n^y \le \delta(\beta_m + 2\alpha \log n^y),$$

or

$$\beta_m - \beta_n \le \delta\left(\frac{\beta_m}{\log n^y} + 2\alpha\right). \tag{3.11}$$

Here the coefficient of δ stays bounded, and by a suitable choice of y we can make δ arbitrarily small by the familiar results on diophantine approximation. Thus the right side of (3.11) can be arbitrarily small, and we conclude that $\beta_m \le \beta_n$. Since the roles of m and n are symmetrical, similarly we obtain $\beta_n \le \beta_m$ and hence $\beta_m = \beta_n$ for all pairs of integers. □

4. Additive functions with monotonic norm

Now we prove Theorem 1.1.

Let f be an additive function with monotonic norm. We know that

$$|f(n)|^2 = \alpha(\log n)^2 + \beta \log n. \tag{4.1}$$

First observe that for every pair of coprime integers we have

$$|f(mn)|^2 = |f(m) + f(n)|^2 = |f(m)|^2 + |f(n)|^2 + 2\big(f(m), f(n)\big),$$

thus

$$\big(f(m),f(n)\big)=\frac{1}{2}\big(|f(mn)|^2-|f(m)|^2-|f(n)|^2\big)=\alpha\log m\log n. \tag{4.2}$$

Now we show that $f(n)/\log n$ converges. For coprime m, n we have

$$\begin{aligned}\left|\frac{f(m)}{\log m}-\frac{f(n)}{\log n}\right|^2&=\frac{|f(m)|^2}{(\log m)^2}+\frac{|f(n)|^2}{(\log n)^2}-2\frac{\big(f(m),f(n)\big)}{\log m\log n}\\&=\beta\left(\frac{1}{\log m}+\frac{1}{\log n}\right);\end{aligned} \tag{4.3}$$

the last equality follows from equation (4.2). Consequently

$$\left|\frac{f(m)}{\log m}-\frac{f(n)}{\log n}\right|\le\frac{c}{\log\min(m,n)}$$

with $c=\sqrt{2\beta}$. (As a byproduct, we get that $\beta\ge 0$.)

To estimate the above difference for any two, not necessarily coprime integers m,n select a prime $p>\max(m,n)$. On applying the above inequality for the pairs (m,p) and (n,p) the triangle inequality gives

$$\left|\frac{f(m)}{\log m}-\frac{f(n)}{\log n}\right|\le\left|\frac{f(m)}{\log m}-\frac{f(p)}{\log p}\right|+\left|\frac{f(p)}{\log p}-\frac{f(n)}{\log n}\right|\le\frac{2c}{\log\min(m,n)}.$$

Hence $f(n)/\log n$ is a Cauchy-sequence, thus the limit $a=\lim f(n)/\log n$ exists. It satisfies

$$|a|^2=\lim_{n\to\infty}\frac{|f(n)|^2}{(\log n)^2}=\alpha.$$

by (4.1).

Write $f(n)=a\log n+g(n)$ (so that $g(n)/\log n\to 0$). Substituting this into (4.3) we see that

$$\left|\frac{g(m)}{\log m}-\frac{g(n)}{\log n}\right|^2=\beta\left(\frac{1}{\log m}+\frac{1}{\log n}\right)$$

for coprime m,n. Now fix n and let $m\to\infty$ through the primes. Since $g(m)/\log m\to 0$, the above formula yields

$$|g(n)|^2=\beta\log n. \tag{4.4}$$

If $\beta = 0$, then this implies $g(n) = 0$, $f(n) = a \log n$ for all n and we are finished.

Assume now $\beta \neq 0$. If $(m, n) = 1$, then we have

$$\big(g(m), g(n)\big) = \frac{1}{2}\big(|g(mn)|^2 - |g(m)|^2 - |g(n)|^2\big) = 0.$$

For a prime p, let E_p be the subspace generated by the values of $g(p^k)$. These subspaces are pairwise orthogonal by the above formula. Equation (1.2) is a particular case of (4.4).

Finally we show that they are all orthogonal to a, which is equivalent to saying that $\big(a, g(n)\big) = 0$ for all n. To see this just observe that

$$\begin{aligned} |f(n)|^2 &= \alpha(\log n)^2 + \beta \log n \\ &= |a \log n + g(n)|^2 = \alpha(\log n)^2 + 2\big(a, g(n)\big) \log n + \beta \log n. \end{aligned}$$

This concludes the proof.

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