

Some conjectures in the theory of exponential diophantine equations

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Dedicated to Professor K. Győry on his 60th birthday

1. Conjecture on a hyperelliptic equation

For integers $a > 0$, $b > 0$ and $k \neq 0$, we recall Pillai's equation

$$(1.1) \quad ax^m - by^n = k$$

in integers $x > 1$, $y > 1$, $m > 1$, $n > 1$ with $mn \geq 6$.

PILLAI [10] conjectured that (1.1) has only finitely many solutions. Now we formulate a conjecture which implies Pillai's Conjecture and a theorem of SCHINZEL and TIJDEMAN [12] that for a polynomial with integer coefficients and at least two distinct roots, there are only finitely many perfect powers in its values at integral points. For this, we introduce some notation. Let α be a rational number written as $\frac{a}{b}$ in its reduced form. We define

$$H(\alpha) = \max(|a|, |b|).$$

We observe that

$$H(\alpha^{-1}) = H(\alpha) \quad \text{for } \alpha \neq 0$$

and

$$(1.2) \quad (H(\alpha))^{-1} \leq |\alpha| \leq H(\alpha) \quad \text{for } \alpha \neq 0.$$

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Let $f(X)$ be a polynomial of degree n with rational coefficients such that it has at least two distinct roots and $f(0) \neq 0$. Let L be the number of non-zero coefficients of f . For non-zero rational numbers

$$b_1, \dots, b_L$$

with

$$n_1 > \dots > n_L, \quad n_1 = n, \quad n_L = 0,$$

let

$$f(X) = b_1 X^{n_1} + \dots + b_{L-1} X^{n_{L-1}} + b_L.$$

Let H be a number satisfying

$$H \geq \max_{1 \leq i \leq L} H(b_i).$$

The right hand side of the above inequality is called the height of f . All the results mentioned in this paper are effective and all the constants appearing in this paper are effectively computable. Now we are ready to state our conjecture.

Conjecture 1.1. *Let $m \geq 2$, and let x and y with $|y| > 1$ be integers satisfying*

$$(1.3) \quad f(x) = y^m.$$

There exists a number C depending only on L and H such that either

$$m \leq C$$

or

$$y^m - f(x) = y^m - b_1 x^{n_1} - \dots - b_{L-1} x^{n_{L-1}} - b_L$$

has a proper subsum which vanishes.

The assumptions that f has at least two distinct roots and $f(0) \neq 0$ are necessary in Conjecture 1.1. For observing this, we take

$$f(X) = X^m, \quad f(2) = 2^m \quad \text{for } m = 2, 3, \dots$$

and

$$f(X) = 4X^{m+1} - 19X^m, \quad f(5) = 5^m \quad \text{for } m = 2, 3, \dots$$

If we consider

$$f(X) = X^m + X - 3, \quad f(3) = 3^m \quad \text{for } m = 2, 3, \dots$$

we see that the possibility of the proper subsum vanishing in Conjecture 1.1 is not ruled out. For positive integers μ, ν with $\mu > \nu$ and $\lambda = (\mu^m - \nu^m)^2$, $x = \mu^m + \nu^m$, the polynomial $f(X) = (X^2 - \lambda)/4$ satisfies $f(x) = (\mu\nu)^m$ for $m \geq 2$. Thus the dependence of C on H in the Conjecture is necessary. For an integer $x > 1$, we consider

$$f(X) = (x-1)(X^{m-1} + \dots + X) + x, \quad f(x) = x^m \quad \text{for } m = 3, 4, \dots$$

in order to observe that the dependence of C on L in the Conjecture is also necessary.

2. Consequences of Conjecture 1.1

Pillai's Conjecture has been confirmed (see [16, Chapter 12]) if at least one of the four variables in (1.1) is fixed. This is also the case if $m = n$ in (1.1). We show

Corollary 2.1. *Conjecture 1.1 implies Pillai's Conjecture.*

PROOF. Suppose that (1.1) is satisfied and Conjecture 1.1 is valid. There is no loss of generality in assuming that $\gcd(a, b, k) = 1$. We rewrite (1.1) as

$$y^n = \frac{a}{b}x^m - \frac{k}{b}.$$

Thus we take

$$f(X) = \frac{a}{b}X^m - \frac{k}{b}$$

in Conjecture 1.1. We observe that $f(0) \neq 0$ since k is non-zero and $f(X)$ has at least two distinct roots since $m \geq 2$. Further

$$L = 2, \quad H = \max(|a|, |b|, |k|)$$

and

$$f(x) = y^n.$$

It is clear that

$$0 = y^n - f(x) = y^n - \frac{a}{b}x^m + \frac{k}{b}$$

has no proper subsum which vanishes. Hence we conclude from Conjecture 1.1 that n is bounded by a number depending only on a , b and k . Similarly, we derive that m is bounded by a number depending only on a , b and k . Now we apply a theorem of BAKER [1] on integral solutions of hyperelliptic equations to (1.1) and we conclude Pillai's Conjecture since $mn \geq 6$. $\square$

As stated in Section 1, SCHINZEL and TIJDEMAN [12] proved

Theorem 2.2. *Let $f(X)$ be a polynomial with rational coefficients and at least two distinct roots. If m , x and y with $m \geq 2$ and $|y| > 1$ are integers satisfying (1.3), then m is bounded by a number depending only on f .*

Corollary 2.3. *Conjecture 1.1 implies Theorem 2.2 unless $f(0) = 0$ and f has at most two distinct roots.*

In fact, we show that Conjecture 1.1 implies that if m , x and y with $m \geq 2$ and $|y| > 1$ are integers satisfying (1.3), then m is bounded by a number depending only on the height of f and the number of non-zero coefficients of f .

PROOF. We assume Conjecture 1.1. First we consider the case that $f(0) \neq 0$. Let m , x and y with $m \geq 2$ and $|y| > 1$ be integers satisfying (1.3). We observe that H depends only on f and $L \leq n = \deg f$. Therefore we see that the constant C appearing in Conjecture 1.1 depends only on f . Further we apply Conjecture 1.1 to suppose that $y^m - f(x)$ has a proper subsum which vanishes. Then we see from (1.3) that its complement is a proper subsum which also vanishes. Thus

$$a_{m_1}x^{m_1} + \cdots + a_{m_t}x^{m_t} = 0,$$

where $m_1 > m_2 > \cdots > m_t$; $a_{m_1}, \dots, a_{m_t}$ are coefficients of f and $a_{m_1} \cdots a_{m_t} \neq 0$. Then

$$a_{m_1}x^{m_1} = -a_{m_2}x^{m_2} - a_{m_3}x^{m_3} - \cdots - a_{m_t}x^{m_t}.$$

Dividing both the sides by x^{m_1-1} , we have

$$a_{m_1}x = -a_{m_2}x^{-(m_1-m_2)+1} - a_{m_3}x^{-(m_1-m_3)+1} - \dots$$

Thus we see from (1.2) that

$$|a_{m_1}x| \leq H\left(1 + \frac{1}{|x|} + \frac{1}{|x|^2} + \dots\right) \leq 2H \quad \text{if } |x| > 1.$$

On the other hand, we observe from (1.2) that

$$|a_{m_1}x| \geq H^{-1}|x|.$$

Hence $|x| \leq 2H^2$. Consequently, we see from (1.3) that $|y|^m$ is bounded by a number depending only on f and this is also the case with m since $|y| > 1$.

Next, we turn to the case $f(0) = 0$. Then we may suppose that f has at least two distinct non-zero roots. We write $f(X) = X^r g(X)$ where $g(0) \neq 0$ and g has at least two distinct non-zero roots. Then we see from (1.3) that there exists a polynomial $g_1(X)$ with at least two distinct non-zero roots and with rational coefficients whose heights are bounded by a number depending only on the height of f , such that $g_1(x)$ is an m -th power of a positive integer greater than 1. Now we apply the previous case to complete the proof of Corollary 2.3. $\square$

3. Generalised $a b c$ Conjecture and Conjecture 1.1

We state the Generalised $a b c$ Conjecture from DARMON and GRANVILLE [4, p. 533].

Generalised $a b c$ Conjecture. *Let $N \geq 3$ and $x_1, \dots, x_N$ be non-zero integers satisfying*

$$x_1 + \dots + x_N = 0, \quad \gcd(x_1, \dots, x_N) = 1$$

and let no proper subsum of $x_1 + \dots + x_N$ vanishes. Then there exist numbers C_1 and C_2 depending only on N such that

$$\max_{1 \leq i \leq N} |x_i| \leq C_1 \left(\prod_{p \mid (x_1 \cdots x_N)} p \right)^{C_2}.$$

Corollary 3.1. *The Generalised $a b c$ Conjecture implies Conjecture 1.1.*

PROOF. The proof of Corollary 3.1 depends on Theorem 2.2. We denote by $C_3, \dots, C_7$ numbers depending only on L and H . We suppose (1.3). By Theorem 2.2, we may assume that $n = \deg f \geq C_3$ with C_3 sufficiently large. Further we may suppose that no proper subsum of

$$y^m - f(x) = y^m - b_1 x^{n_1} - \dots - b_{L-1} x^{n_{L-1}} - b_L = 0$$

vanishes. Now we clear out the denominators of the rational numbers b_i in the above relation and then we divide both sides by the greatest common divisor of the terms. We observe that the greatest common divisor is bounded since a_0 is non-zero. Now we apply the Generalised $a b c$ Conjecture to conclude that

$$|y|^m \leq C_4 (|yx|)^{C_5}.$$

Further we see from (1.3) that

$$|x|^n \leq C_6 |y|^m.$$

By taking $C_3 > 2C_5$, we get

$$|x|^{2C_5} \leq |x|^n \leq C_6 |y|^m.$$

Consequently

$$|y|^{m/2} < C_4 C_6^{1/2} |y|^{C_5}$$

which implies that $m \leq C_7$ since $|y| > 1$. This completes the proof of Corollary 3.1. $\square$

4. Problems on an equation of Nagell–Ljunggren

We consider the following equation:

$$(4.1) \quad \frac{x^m - 1}{x - 1} = y^q \quad \text{in integers } x > 1, y > 1, m > 2, q \geq 2.$$

By writing $y^q = (y^{q/p})^p$, there is no loss of generality in assuming that q is prime in (4.1). We observe that

$$(4.2) \quad \frac{3^5 - 1}{3 - 1} = 11^2, \quad \frac{7^4 - 1}{7 - 1} = 20^2, \quad \frac{18^3 - 1}{18 - 1} = 7^3.$$

The initial contributions on (4.1) are due to Nagell–Ljunggren and therefore, we call (4.1) the equation of Nagell–Ljunggren. LJUNGGREN [8] proved that (4.1) with $q = 2$ has no solution other than the ones given by (4.2). Therefore, we suppose from now on that $q > 2$ in (4.1). Further it follows from the results of NAGELL [9] and LJUNGGREN [8] that (4.1) implies

$$m \equiv 5 \pmod{6} \text{ if } q = 3 \text{ and } 3 \nmid m, \quad 4 \nmid m$$

unless $(x, y, m, q) = (18, 7, 3, 3)$. For a survey on (4.1), we refer to SHOREY and TIJDEMAN [16, Chapter 12] and SHOREY [15, Section 4].

Let $\nu > 1$ be an integer. Let $P(\nu)$ denote the greatest prime factor of ν . We write $\omega(\nu)$ and $Q(\nu)$ for the number of distinct prime divisors of ν and the greatest square-free factor of ν , respectively. We recall that $\varphi(\nu)$ is the number of positive integers less than ν and coprime to ν . We start with the following factorisation on (4.1) given by SHOREY [13].

Lemma 4.1. *Assume (4.1). Let D be a positive divisor of m such that*

$$\gcd(D, m/D) = \gcd(D, \varphi(Q(m/D))) = 1.$$

Then

$$\frac{(x^D)^{m/D} - 1}{x^D - 1} = y_1^q, \quad \frac{x^D - 1}{x - 1} = y_2^q$$

for positive integers y_1 and y_2 .

Our final aim is to prove on (4.1) the following

Conjecture 4.2. *Equation (4.1) has no solution other than the ones given by (4.2).*

A weaker version of Conjecture 4.2 states

Conjecture 4.3. *Equation (4.1) has only finitely many solutions.*

Let $m = P_1^{A_1} \cdots P_s^{A_s}$ where $P_1 < \cdots < P_s$ are prime numbers and $A_1, \dots, A_s$ are positive integers. We apply Lemma 4.1 successively with $D = P_s^{A_s}, \dots, D = P_2^{A_2}$ to derive

Corollary 4.4. *It suffices to prove Conjecture 4.3 for $\omega(m) = 1$.*

Thus the case $\omega(m) = 1$ is the most difficult part of Conjecture 4.3. But we do not know an answer even to the following simpler question.

Conjecture 4.6. *Equation (4.1) with $\omega(m) \geq 2$ has only finitely many solutions.*

Another conjecture lying between Conjectures 4.3 and 4.6 states

Conjecture 4.5. *Equation (4.1) has only finitely many solutions whenever x is a perfect power.*

Conjecture 4.2 implies Conjecture 4.3 which gives Conjecture 4.5. Now we show

Corollary 4.7. *Conjecture 4.5 implies Conjecture 4.6.*

PROOF. Assume (4.1) and Conjecture 4.5. Let $m = P_1^{A_1} \cdots P_s^{A_s}$ as above with $s \geq 2$. Then we apply Lemma 4.1 with $D = P_s^{A_s}$ to suppose that $m = 2D$ and

$$x^D + 1 = y_1^q.$$

This is Catalan's equation and TIJDEMAN [17] proved that it has only finitely many solutions. This completes the proof of Corollary 4.7. $\square$

There has been progress on Conjecture 4.5 recently. SARADHA and SHOREY [11] confirmed the conjecture when x is a square. In fact they proved that (4.1) has no solution whenever $x = z^2$ with $z \geq 32$ and $z \in \{2, 3, 4, 8, 9, 16, 25, 27\}$. Further BENNETT [2] and BUGEAUD, MIGNOTTE, ROY and SHOREY [3], independently, covered the remaining cases. Thus (4.1) has no solution if x is a square. Further HIRATA-KOHNO and SHOREY [6] confirmed the conjecture when $x = z^\mu$ where μ is a fixed odd prime and $q > 2(\mu - 1)(2\mu - 3)$. By taking $\mu = 3$ in the preceding result, we see that (4.1) with $x = z^3$ and $q \notin \{5, 7, 11\}$ has only finitely many solutions. For a survey of results on Conjecture 4.5, we refer to SHOREY [15, Section 4].

5. Results on Conjecture 4.6

A weaker version of Conjecture 4.6, namely that (4.1) with $\omega(m) > q - 2$ has only finitely many solutions, has been given by SHOREY [13], [14]. The proof depends on the results of SHOREY [13, [14] that (4.1) has only finitely many solutions if either $m \equiv 1 \pmod{q}$ or x is a q -th power. These results have been improved as follows:

Lemma 5.1. *Equation (4.1) has no solution whenever x is a q -th power.*

Lemma 5.2. *Equation (4.1) with $m \equiv 1 \pmod{q}$ has no solution.*

Lemma 5.1 is due to LE [7] and Lemma 5.2 is an immediate consequence of a theorem of BENNETT [2] saying that for a positive integer a , the equation

$$(a+1)x^n - ay^n = 1 \text{ has no solution in integers } x > 1, y > 1, n \geq 3.$$

We use the above lemmas in the proof of Shorey's result saying that (4.1) with $\omega(m) > q-2$ has only finitely many solutions, to show

Theorem 5.3. *Equation (4.1) with $\omega(m) > q-2$ has no solution.*

PROOF. Suppose that (4.1) is satisfied. We write

$$m = q^e p_1^{a_1} \cdots p_r^{a_r}$$

where $e \geq 0$, $a_1 > 0, \dots, a_r > 0$ and $p_1 < p_2 < \cdots < p_r$ are prime numbers different from q . For $1 \leq \mu \leq \nu \leq r$, we put

$$m_{\mu,\nu} = p_\mu^{a_\mu} \cdots p_\nu^{a_\nu}.$$

By repeated application of Lemmas 4.1 and 5.2, we derive that none of $p_1, \dots, p_r$ is congruent to 1 (mod q). Then we apply Lemma 4.1 with $D = q^e$ and Lemma 5.1 to conclude that $e = 0$. For $1 \leq \mu \leq \nu \leq r$, we write $D_1 = m_{1,\mu-1}$, $D_2 = m_{\mu,\nu}$ and $D_3 = m_{\nu+1,r}$. We apply Lemma 4.1 with $D = D_3$ and $D = D_2$ to derive that $\frac{X^{D_2}-1}{X-1}$ with $X = x^{D_3}$ is a q -th power. Then we conclude from Lemma 5.2 that none of $m_{\mu,\nu}$ with $1 \leq \mu \leq \nu \leq r$ is congruent to 1 (mod q). Finally, we consider

$$m_{1,1} = p_1^{a_1}, m_{1,2} = p_1^{a_1} p_2^{a_2}, \dots, m_{1,r} = p_1^{a_1} \cdots p_r^{a_r}.$$

We know that none of these is congruent to 0, 1 (mod q). Further, for $1 \leq \mu < \nu \leq r$ we observe that $m_{1,\mu}$ and $m_{1,\nu}$ are incongruent (mod q), otherwise

$$\frac{m_{1,\nu}}{m_{1,\mu}} = m_{\mu+1,\nu} \equiv 1 \pmod{q}.$$

Hence $\omega(m) = r \leq q-2$. This completes the proof of Theorem 5.3. $\square$

Now we consider (4.1) with the additional assumption

$$(5.1) \quad \gcd(m, \varphi(Q(m))) = 1.$$

ERDŐS [5] gave an asymptotic formula for the number of positive integers satisfying (5.1). Thus there are infinitely many positive integers m satisfying (5.1). The assumption $\omega(m) > q - 2$ in the above results can be relaxed in this case. SHOREY [14] showed that (4.1) with (5.1) and

$$(5.2) \quad 2^{\omega(m)} > q - 1$$

has only finitely many solutions. In fact, we have

Theorem 5.4. *Equation (4.1) with (5.1) and (5.2) has no solution.*

The proof depends on Lemmas 4.1, 5.2 and a result of LE [7]. The derivation of Theorem 5.4 from these results is similar to that of Theorem 5.3 from Lemmas 4.1, 5.1, 5.2 and we refer to SHOREY [14] for details.

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