

Geometry of multiparametrized Lagrangians

By MIHAI ANASTASIEI (Iași) and HIROAKI KAWAGUCHI* (Fujisawa)

Dedicated to Professor Lajos Tamásy on his 70th birthday

1. Introduction

There are many problems in theoretical physics and variational calculus in which the Euler–Lagrange equations

$$(1.1) \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial y^i} \right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0,$$

where $y^i = \frac{dx^i}{dt}$, are fundamental.

The function $\mathcal{L}$ depending on (x^i) and (y^i) , $i = 1, 2, \dots, n$, is called a Lagrangian function or simply a Lagrangian. From a geometrical point of view a Lagrangian (of M) is a function $\mathcal{L} : TM \rightarrow R$, where TM is the total space of the tangent bundle (TM, τ, M) to a smooth (C^∞) manifold M . A point $v \in TM$ has the local coordinates (x^i, y^i) , where (x^i) are the local coordinates of $x = \tau(v)$, and $v_x = y^i \left(\frac{\partial}{\partial x^i} \right)_x$. Thus

$$\mathcal{L} : (x^i, y^i) \rightarrow \mathcal{L}(x^i, y^i), \quad i = 1, 2, \dots, n = \dim M.$$

Expanding the time derivative, eq. (1.1) becomes

$$(1.2) \quad \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} \frac{d^2 x^j}{dt^2} = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial y^i \partial x^j} \frac{dx^j}{dt}$$

and in order to put it in a normal form we must assume

$$(1.3) \quad \det \left(\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} \right) \neq 0.$$

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If (1.3) holds, $\mathcal{L}$ is called a regular Lagrangian.

If we put $\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial x^i} dx^i$, then $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$ gives a symplectic two forms on TM which is nondegenerate if $\mathcal{L}$ is a regular Lagrangian. Thus the results of symplectic geometry may be applied.

A different point of view was proposed by R. Miron. Namely, using $g_{ij} = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}$ and a nonlinear connection on TM , a metrical structure on TM is introduced so that the results of the so-called Lagrange geometry may be applied.

The origin of Lagrange geometry is as follows. It is usual to call the pair $(M, \mathcal{L})$, where $\mathcal{L}$ is a regular Lagrangian, a Lagrange space. If (M, F) is a Finsler space (cf. M. MATSUMOTO [5]) then taking $\mathcal{L} = F^2$ we see that any Finsler space is a Lagrange space. As is well-known Finsler geometry has a long tradition and its body of results is very large.

On the other hand, as Professor RADU MIRON showed in [6] and [7], the main results of Finsler geometry regarding the nonlinear connection, the Cartan connection and so on can be extended to Lagrange spaces. Thus the geometry of the pair $(M, \mathcal{L})$, called Lagrange geometry, was developed in the last ten years.

In dynamics as well as in variational calculus (see R. HERMANN [4] p.117) time dependent Lagrangians are also considered.

A regular time dependent Lagrangian is a function $\mathcal{L} : TM \times R \rightarrow R$, $(x^i, y^i, t) \rightarrow L(x^i, y^i, t)$ such that $\det \left(\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} \right) \neq 0$. We developed a geometrical theory of such Lagrangians in some recent papers ([1]–[3]). Now we consider multiparametrized Lagrangians i.e. functions $\mathcal{L} : TM \times R^m \rightarrow R$, $(x^i, y^i, t^1, \dots, t^m) \rightarrow \mathcal{L}(x^i, y^i, t^1, \dots, t^m)$. Such Lagrangians appear in variational problems for which the constraints are considered (cf. R. HERMANN [4]).

We shall derive here the main facts from the geometry of the manifold $E = TM \times R^m$ fibered over M , endowed with a regular multiparametrized Lagrangian $\mathcal{L}$, i.e. $\det \left(\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} \right) \neq 0$ be assumed.

2. On the manifold $TM \times R^m$ fibered over M

A transformation $(x^i, y^i, t^\alpha) \rightarrow (\bar{x}^i, \bar{y}^i, \bar{t}^\alpha)$ of local coordinates on $TM \times R^m$ is of the form

$$(2.1) \quad \begin{aligned} \bar{x}^i &= \bar{x}^i(x^1, \dots, x^n), \quad \text{rank} \left(\frac{\partial \bar{x}^i}{\partial x^j} \right) = n \\ \bar{y}^i &= \frac{\partial \bar{x}^i}{\partial x^j} y^j, \quad \bar{t}^\alpha = t^\alpha. \end{aligned}$$

Here and in the sequel the indices $i, j, \dots$, run from 1 to $n = \dim M$ and $\alpha, \beta, \gamma, \dots$, run from 1 to m .

Sometimes we shall set $(y^i, t^\alpha) = z^a$, $a, b, c, \dots = 1, \dots, n + m$ and then (2.1) becomes

$$(2.2.) \quad \begin{aligned} \bar{x}^i &= \bar{x}^i(x^1, \dots, x^n), \quad \text{rank} \left(\frac{\partial \bar{x}^i}{\partial x^j} \right) = n \\ \bar{z}^a &= M_b^a(x) z^b, \quad \text{rank}(M_b^a) = n + m. \end{aligned}$$

Computing the Jacobian matrix of the mapping (2.2) we obtain

Theorem 2.1. *The manifold $TM \times R^m$ is orientable.*

The triad $(TM \times R^m, \pi, M)$, $\pi(v, t^\alpha) = \tau(v)$ is a vector bundle of rank $n + m$ (the local fibre is $T_x M \times R^m$, at $x = \tau(v)$).

By a general result on vector bundles (R. MIRON and M. ANASTASIEI [7]) we have

Theorem 2.2. *If M is paracompact then $TM \times R^m$ is paracompact.*

Thus if M is paracompact then $TM \times R^m$ admits smooth partitions of unity.

Let us set $E = TM \times R^m$ and let $\pi^T : TE \rightarrow TM$ be the tangent mapping of π . Then $VE = \ker \pi^T$ is a vector subbundle of the tangent bundle (TE, τ_E, E) to E . We call it the vertical bundle over E . The natural basis in $T_u E$, $u \in E$, is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t^\alpha} \right)$ and $\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial t^\alpha} \right)$ gives a basis of $V_u E$.

These bases transform under (2.1) as follows:

$$(2.3) \quad \begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial^2 \bar{x}^h}{\partial x^i \partial x^j} y^j \frac{\partial}{\partial \bar{y}^h} + \frac{\partial \bar{x}^h}{\partial x^i} \frac{\partial}{\partial \bar{x}^h} \\ \frac{\partial}{\partial y^i} &= \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial}{\partial \bar{y}^k}, \quad \frac{\partial}{\partial t^\alpha} = \frac{\partial}{\partial \bar{t}^\alpha}. \end{aligned}$$

By (2.3) it comes out that $J : T_u E \rightarrow T_u E$ given by

$$(2.4) \quad J \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^i}, \quad J \left(\frac{\partial}{\partial z^a} \right) = 0 \quad \text{is well-defined.}$$

By a direct calculation one gets

Theorem 2.3.

$$\begin{aligned} (i) \quad J^2 &= 0, & (i) \quad \text{Ker } J &= VE, \quad \text{Im } J \subset VE, \\ (iii) \quad N_J &= 0, & (iv) \quad L_C J &= -J. \end{aligned}$$

Here N_J denotes the Nijenhuis tensor field associated to J and $L_C J$ denotes the Lie derivative of J with respect to the Liouville vector field

$$C = z^a \frac{\partial}{\partial z^a} :$$

$$(L_C J)(X) = [C, JX] - J[C, X], \quad X \in \mathfrak{X}(E).$$

Thus, the manifold $TM \times R^m$ is endowed with an almost tangent structure (cf. (i)) which is integrable (cf. (iii)) and homogeneous of degree 0 (cf. (iv)).

3. Nonlinear connections on $E = TM \times R^m$

Now we shall regard the vertical bundle over E as a distribution $u \rightarrow V_u E$, $u \in E$, on E .

Definition 3.1. A nonlinear connection on E is a distribution $u \rightarrow H_u E$ which is supplementary to the vertical distribution on E , that is,

$$(3.1) \quad T_u E = H_u E \oplus V_u E, \quad u \in E$$

holds good.

Locally, the distribution $u \rightarrow H_u E$, called horizontal distribution, is completely determined by n local vector fields

$$(3.2) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial z^a} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} - N_i^\alpha \frac{\partial}{\partial t^\alpha}.$$

The form of these vector fields is a consequence of the fact that $\pi^T|_{HE}$ is an isomorphism which carries $\frac{\delta}{\delta x^i}$ to $\frac{\partial}{\partial x^i}$. Since $u \rightarrow H_u E$ is a global distribution, the adapted frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial z^a})$ belonging to the decomposition (3.1) must transform under (2.1) as follows:

$$(3.3) \quad \frac{\delta}{\delta x^i} = \frac{\partial \bar{x}^h}{\partial x^i} \frac{\delta}{\delta \bar{x}^h}, \quad \frac{\partial}{\partial z^a} = M_a^b(x) \frac{\partial}{\partial \bar{z}^b}.$$

By (3.3), the local coefficients $(N_i^a(x, y, t))$ have the following transformation law under (2.1):

$$(3.4) \quad N_i^a \frac{\partial \bar{x}^i}{\partial x^j} = M_b^a N_j^b - \frac{\partial M_b^a}{\partial x^j} z^b.$$

Taking in (3.4) $a = h$ and then $a = \alpha$ we see that (3.4) is equivalent to:

$$(3.5) \quad \begin{aligned} \bar{N}_i^h \frac{\partial \bar{x}^i}{\partial x^j} &= \frac{\partial \bar{x}^h}{\partial x^k} N_j^k - \frac{\partial^2 \bar{x}^h}{\partial x^j \partial x^k} y^k, \\ \bar{N}_i^\alpha \frac{\partial \bar{x}^i}{\partial x^j} &= N_j^\alpha. \end{aligned}$$

Thus $(N_i^j(x, y, t))$ transform as the coefficients of a nonlinear connection on TM and (N_i^α) as the components of a covector on M . We shall say that (N_i^α) defines an M -covector on E for every α .

Conversely, a set of functions $(N_i^j(x, y, t), N_i^\alpha(x, y, t))$ which satisfy (3.5) defines a nonlinear connection on E .

The following theorem says us that a regular multiparametrized Lagrangian $\mathcal{L}$ determines a nonlinear connection on E .

Let us put:

$$(3.6) \quad g_{ij}(x, y, t^\alpha) = \frac{1}{2} \frac{\partial^2 \mathcal{L}(x, y, t^\alpha)}{\partial y^i \partial y^j}$$

$$(3.7) \quad G^i(x, y, t^\alpha) = \frac{1}{4} g^{ik} \left(\frac{\partial^2 \mathcal{L}}{\partial y^k \partial x^j} y^j - \frac{\partial \mathcal{L}}{\partial x^k} \right).$$

Theorem 3.1. *The set of functions*

$$N_j^i(x, y, t^\alpha) = \frac{\partial G^i(x, y, t^\alpha)}{\partial y^j}; \quad N_i^\alpha = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial t^\alpha \partial y^i}$$

defines a nonlinear connection on $TM \times R^m$.

PROOF. One checks by a tedious computation that these functions transform under (2.1) as in (3.5).

Remark 3.1. The form of G^i in (3.7) was suggested by the form (1.2) of the Euler-Lagrange equations.

Remark 3.2. The nonlinear connection determined by $\mathcal{L}$ is symmetric in the sense that its torsion

$$t_{ij}^k = \frac{\partial N_i^k}{\partial y^j} - \frac{\partial N_j^k}{\partial y^i}$$

vanishes.

4. Other geometrical structures on $TM \times R^m$

Let $N_{\mathcal{L}}$ be the nonlinear connection on E defined by $\mathcal{L}$ and $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial z^a})$ the corresponding adapted frame.

If we set:

$$(4.1) \quad P \left(\frac{\delta}{\delta x^i} \right) = \frac{\delta}{\delta x^i}, \quad P \left(\frac{\partial}{\partial z^a} \right) = -\frac{\partial}{\partial z^a},$$

we obtain an almost product structure on E , that is, $P^2 = I$, where I denotes the Kronecker tensor field.

Now let $(dx^i, \delta z^a)$ be the frame dual to $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial z^a})$. It follows that $\delta z^a = dz^a + N_i^a dx^i$, or equivalently,

$$(4.2) \quad \begin{aligned} \delta y^i &= dy^i + N_k^i dx^k \\ \delta t^\alpha &= dt^\alpha + N_i^\alpha dx^i. \end{aligned}$$

Let us define a linear mapping $F : T_u E \rightarrow T_u E$ by

$$(4.3) \quad F \left(\frac{\delta}{\delta x^i} \right) = -\frac{\partial}{\partial y^i}, \quad F \left(\frac{\partial}{\partial y^i} \right) = \frac{\delta}{\delta x^i}, \quad F \left(\frac{\partial}{\partial t^\alpha} \right) = 0.$$

We immediately get

Theorem 4.1. *The following equalities hold good:*

- (i) $\text{rank } F = 2n$,
- (ii) $F^3 + F = 0$,
- (iii) $F^2 = -I + \frac{\partial}{\partial t^\alpha} \otimes \delta t^\alpha$

Thus $TM \times R^m$ is framed manifold. The frame structure $(F, \frac{\partial}{\partial t^\alpha}, \delta t^\alpha)$ is said to be normal if the tensor field

$$(4.5) \quad S(X, Y) = N_F(X, Y) + d(\delta t^\alpha)(X, Y) \frac{\partial}{\partial t^\alpha}, \quad X, Y \in \mathfrak{X}(E)$$

vanishes identically.

A computation in local coordinates leads to

Theorem 4.2. *The frame structure $(F, \frac{\partial}{\partial t^\alpha}, \delta t^\alpha)$ is normal if and only if*

- 1) *The curvature of $N_{\mathcal{L}}$ vanishes i.e.*

$$\Omega_{ij}^q := \frac{\delta N_j^a}{\delta x^i} - \frac{\delta N_i^a}{\delta x^j} = 0,$$

- 2) $\frac{\partial N_i^a}{\partial t^\alpha} = 0$.

It is obvious that the following tensor field

$$(4.6) \quad G = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j + \sum_{\alpha=1}^m \delta t^\alpha \otimes \delta t^\alpha$$

defines a metric structure on $TM \times R^m$.

It is Riemannian if the quadratic form $g_{ij} \xi^i \xi^j$, $(\xi^i) \in R^n$, is positive definite.

By (4.6) we have

Theorem 4.3. *The horizontal and vertical distributions are orthogonal with respect to G .*

Some computations in local coordinates give

Theorem 4.4. *The following equations hold good:*

- (i)
$$G(FX, FY) = G(X, Y) - \sum_{\alpha=1}^m \delta t^\alpha(X) \delta t^\alpha(Y),$$
- (ii)
$$\delta t^\alpha(X) = G\left(\frac{\partial}{\partial t^\alpha}, X\right),$$
- (iii)
$$G(PX, Y) = G(X, PY), \quad X, Y \in \mathfrak{X}(E).$$

Thus, the manifold $TM \times R^m$ possesses an almost product structure, a frame structure and a metric structure related by (i)–(iii) in Theorem 4.4.

5. M -connections on $E = TM \times R^m$

We identify, as is usual, a linear connection on E with the operator of covariant derivative D associated to it.

Definition 5.1. A linear connection D on E is said to be an M -connection if the following conditions hold good:

- (i) $DP = 0$,
- (ii) $DF = 0$,
- (iii) $D\left(\frac{\partial}{\partial t^\alpha}\right) = 0, \quad \alpha = 1, 2, \dots, m.$

Remark 5.1. An M -connection D on E satisfies also $D(\delta t^\alpha) = 0$ by virtue (iii) in Theorem 4.1.

We have

Theorem 5.1. *A linear connection D on E is an M -connection if and only if in the frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t^\alpha}\right)$ we have*

$$(5.1) \quad \begin{aligned} D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} &= L_{jk}^i \frac{\delta}{\delta x^i}, & D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} &= L_{jk}^i \frac{\partial}{\partial y^i}, \\ D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} &= C_{jk}^i \frac{\delta}{\delta x^i}, & D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} &= C_{jk}^i \frac{\partial}{\partial y^i}, \\ D_{\frac{\delta}{\delta t^\alpha}} \frac{\delta}{\delta x^j} &= C_{j\alpha}^i \frac{\delta}{\delta x^i}, & D_{\frac{\delta}{\delta t^\alpha}} \frac{\partial}{\partial y^j} &= C_{j\alpha}^i \frac{\partial}{\partial y^i} \end{aligned}$$

where, under (2.1), L_{jk}^i change like the coefficients of a linear connection on M , C_{jk}^i change like the components of a tensor field of type (1,2) on M

and $C_{j\alpha}^i$ ($\alpha = 1, 2, \dots, m$) change like the components of a tensor field of type (1,1) on M .

Thus we can give an M -connection as a set of local coefficients $D\Gamma = (L_{jk}^i, C_{jk}^i, C_{j\alpha}^i)$.

Using these coefficients, h - and v -covariant derivatives denoted by short and long horizontal bars, respectively, of any M -tensor can be considered. For instance,

$$\begin{aligned} t_{ij}|_k &= \frac{\delta t_{ij}}{\delta x^k} - L_{ik}^h t_{hj} - L_{jk}^h t_{ih}, \\ t_{ij}|_k &= \frac{\partial t_{ij}}{\partial y^k} - C_{ik}^h t_{hj} - C_{jk}^h t_{ih}, \\ t_{ij}|\alpha &= \frac{\partial t_{ij}}{\partial t^\alpha} - C_{i\alpha}^h t_{hj} - C_{j\alpha}^h t_{ih}. \end{aligned}$$

An M -connection on E is said to be metrical if $DG = 0$. A direct computation gives

Theorem 5.2. *An M -connection is metrical if and only if*

$$(5.2) \quad g_{ij}|_k = 0, \quad g_{ij}|_k = 0, \quad g_{ij}|\alpha = 0,$$

holds.

Let us set $T_{jk}^i = L_{jk}^i - L_{kj}^i$, $S_{jk}^i = C_{jk}^i - C_{kj}^i$. These tensor fields are the torsions of the M -connection ΓD .

On the existence of the metrical M -connection we have the following

Theorem 5.3. *There exists a set of metrical M -connections with $T_{jk}^i = S_{jk}^i = 0$. Their local coefficients are as follows:*

$$(5.3) \quad \begin{aligned} L_{ij}^k &= \frac{1}{2} g^{kh} \left(\frac{\delta g_{hj}}{\delta x^i} + \frac{\delta g_{ih}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^h} \right) \\ C_{ij}^k &= \frac{1}{2} g^{kh} \left(\frac{\partial g_{hj}}{\partial y^i} + \frac{\partial g_{ih}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^h} \right) \\ C_{i\alpha}^k &= \frac{1}{2} g^{kh} \frac{\partial g_{ih}}{\partial t^\alpha} + O_{ih}^{jk} X_{j\alpha}^h, \end{aligned}$$

where $X_{j\alpha}^h$ is an arbitrary M -tensor field of type (1,1) and $O_{ih}^{jk} = \frac{1}{2} (\delta_i^j \delta_h^k - g_{ih} g^{kj})$ is the Obata operator.

PROOF. The condition $g_{ij}|_k = 0$ is equivalent to $\frac{\delta g_{ij}}{\delta x^k} = L_{ik}^h g_{hj} + L_{jk}^h g_{ih}$. Subtracting this from the sum of the other two equations obtained by a cyclic permutation in it of the indices i, j, k and using $T_{jk}^i = 0$ one gets

L_{jk}^i . C_{jh}^i is derived in a similar way. Then it is easy to check that $\frac{1}{2}g^{kh}\frac{\partial g_{ih}}{\partial t^\alpha}$ verifies $g_{ij|\alpha} = 0$. If $C_{i\alpha}^k$ is another solution of the equation $g_{ij|\alpha} = 0$, then $B_{i\alpha}^k = C_{i\alpha}^k - \frac{1}{2}g^{kh}\frac{\partial g_{ih}}{\partial t^\alpha}$ satisfies the equation $g_{ki}B_j^k + g_{jk}B_i^k = 0$.

Using the Obata operator we find that the general solution of the last equation is $B_{i\alpha}^k = O_{ih}^{jk}X_{j\alpha}^h$. $\square$

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MIHAI ANASTASIEI
FACULTATEA DE MATEMATICA
UNIVERSITATEA „AL. I. CUZA”
6600 IAȘI, ROMANIA

HIROAKI KAWAGUCHI
DEPARTMENT OF INFORMATION SCIENCE
SHŌNAN INSTITUTE OF TECHNOLOGY
TSUJIDO, FUJISAWA 251, JAPAN

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