

On a problem of Erdős–Turán

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Abstract. We find a class of real functions f having the property that the inequality $f(p_{n+1}) - 2f(p_n) + f(p_{n-1}) > 0$ holds for infinitely many positive integers n , and $f(p_{n-1}) - 2f(p_n) + f(p_{n+1}) < 0$ holds for infinitely many n .

One interesting question about the properties of the sequence $(p_n)_{n \geq 1}$ of prime numbers is:

Does there exist a positive integer n_0 such that this sequence is convex or concave for all $n > n_0$?

The answer is negative and the proof of this result is given in the paper [1] by P. ERDŐS and P. TURÁN. They established that:

For infinitely many n one has

$$p_{n+1} - 2p_n + p_{n-1} > 0,$$

and for infinitely many n :

$$p_{n+1} - 2p_n + p_{n-1} < 0.$$

C. POMERANCE proved in [2], that there are infinitely many n for which:

$$2p_n < p_{n-i} + p_{n+i} \quad \text{for all } i, \ 1 \leq i \leq n-1.$$

In [3] ERDŐS proved that if

$$\begin{aligned} k \geq 3, \ a_1 + a_2 + \cdots + a_k = 0, \ a_k \neq 0 \quad \text{and} \\ (k-1)a_1 + (k-2)a_2 + \cdots + a_{k-1} = 0, \end{aligned}$$

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then the sequence $x_n = a_1 p_n + a_2 p_{n-1} + \cdots + a_k p_{n+k-1}$ does not keep a constant sign.

Moreover, for fixed $k \neq 0$, $p_{n+1}^k - 2p_n^k + p_{n-1}^k > 0$ for infinitely many n and, also, $p_{n+1}^k - 2p_n^k + p_{n-1}^k < 0$ for infinitely many n .

This means that for $f(x) = x^k$, $k \neq 0$ the sequence $(f(p_n))_{n \geq 1}$ is neither convex nor concave.

We shall say that a function $f : [a_f, \infty) \rightarrow \mathbb{R}$ has property (P) if: For infinitely many n one has

$$f(p_{n+1}) - 2f(p_n) + f(p_{n-1}) > 0,$$

and for infinitely many n :

$$f(p_{n+1}) - 2f(p_n) + f(p_{n-1}) < 0.$$

To find necessary and sufficient conditions for f to have property (P) can be a difficult task.

Let $f(x) = a^x$. In case $a = 2$ one can prove immediately that the sequence is convex, so f does not have the (P) property. In case $a = 1.2$, to prove that f has the (P) property is the same as to prove that there exists an infinity of primes p for which $p+2$ too is prime. Consequently, this would amount to solving one of the greatest open problems in number theory. This example shows that to find a complete answer to our question is a very difficult task indeed. We shall restrict ourselves to finding a sufficient condition and for this purpose we shall consider a class of functions useful in this direction.

For every $f : [a_f, \infty) \rightarrow (0, \infty)$, $f \in C^1$ define $\phi : [a_f, \infty) \rightarrow \mathbb{R}$ by

$$\phi(x) = \frac{xf'(x)}{f(x)}.$$

Let $F = \{f : [a_f, \infty) \rightarrow (0, \infty), f \in C^1, \lim_{x \rightarrow \infty} \phi(x) = k \in \mathbb{R} \setminus \{0\}\}$.

One can prove that in the neighborhood of ∞ the functions of F behave somehow similarly to ax^n , because for $n > k$ one can prove that $\lim_{x \rightarrow \infty} \frac{f(x)}{x^n} = 0$, and for $n < k$, $\lim_{x \rightarrow \infty} \frac{|f(x)|}{x^n} = \infty$.

One can also notice that for real α, β , $\alpha \neq 0$ one has $f \in F$ if and only if $f^\alpha(x) \log^\beta x \in F$.

The main result of our paper is

Theorem 1. *If $f \in F$, then f has the (P) property. Before proving the theorem we shall need some preliminaries. One knows that*

- (1) $p_n \sim n \log n$;
- (2) $\liminf_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{\log p_n} < 0.46665$ (see [4]);
- (3) $\limsup_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{\log p_n} = \infty$ (see [5]).

On the basis of the relations (2) or (3) and of Lemma 3, we now prove our theorem.

Lemma 1. *Let $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ be sequences of positive integers, $\lim_{n \rightarrow \infty} b_n = \infty$. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ then $\lim_{n \rightarrow \infty} \frac{p_{a_n}}{p_{b_n}} = c$.*

PROOF. Let $y_n = \frac{a_n}{b_n}$ so $\lim_{n \rightarrow \infty} y_n = c$. Taking into account (1) it follows that

$$\lim_{n \rightarrow \infty} \frac{p_{a_n}}{p_{b_n}} = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} \cdot \frac{\log a_n}{\log b_n} = c \lim_{n \rightarrow \infty} \left(1 + \frac{\log y_n}{\log b_n} \right) = c.$$

Put $F(n) = f(p_n)$. In case $f \in F$ and $\lim_{x \rightarrow \infty} \phi(x) = k$, we have

Lemma 2. *Under the conditions of Lemma 1, one has*

$$\lim_{n \rightarrow \infty} \frac{F(a_n)}{F(b_n)} = c.$$

PROOF. One has

$$f(x) = f(x_0) \exp \int_{x_0}^x \frac{\phi(t)}{t} dt = f(x_0) \exp \int_{\log x_0}^{\log x} \phi(e^u) du.$$

$$\frac{F(a_n)}{F(b_n)} = \exp \int_{\log b_n}^{\log a_n} \phi(e^u) du = \exp((\log p_{a_n} - \log p_{b_n}) \phi(\theta_n)),$$

where $\min(\log p_{a_n}, \log p_{b_n}) < \theta_n < \max(\log p_{a_n}, \log p_{b_n})$ hence

$\lim_{n \rightarrow \infty} \phi(\theta_n) = k$. It follows that $\lim_{n \rightarrow \infty} \frac{F(a_n)}{F(b_n)} = \exp k \log \left(\frac{p_{a_n}}{p_{b_n}} \right) = c^k$.

Under the conditions of the previous lemmas and with the same notations one has

Lemma 3. *If the sequence $(F(n))_{n \geq 1}$ is convex, then*

$$F(n+1) - F(n) \sim k \frac{F(n)}{n}.$$

PROOF. If the sequence $(F(n))_{n \geq 1}$ is convex it follows that for $m > n > p$:

$$(4) \quad \frac{F(m) - F(n)}{m - n} \geq F(n+1) - F(n) \geq \frac{F(n) - F(p)}{n - p}.$$

For fixed $1 > \delta > 0$, put $m = [(1 + \delta)n]$ and $p = [(1 - \delta)n]$. It follows that $m - n \sim \delta n$, $n - p \sim \delta n$.

Using Lemma 2 one obtains

$$F(m) \sim F(n)(1 + \delta)^k \quad \text{and} \quad F(p) \sim F(n)(1 - \delta)^k,$$

hence

$$\frac{F(m) - F(n)}{m - n} \sim \frac{F(n)((1 + \delta)^k - 1)}{n\delta}.$$

In the same way $\frac{F(n) - F(p)}{n - p} \sim \frac{F(n)(1 - (1 - \delta)^k)}{n\delta}$. Taking into account that $\lim_{\delta \rightarrow 0} \frac{(1 + \delta) - 1}{\delta} = \lim_{\delta \rightarrow 0} \frac{1 - (1 - \delta)^k}{\delta} = k$ and (4), the proof is finished.

PROOF of Theorem 1. Let $(F(n))_{n \geq 1}$ be convex. One has $F(n+1) - F(n) = f(p_{n+1}) - f(p_n) = (p_{n+1} - p_n)f'(\theta_n)$, $p_n < \theta_n < p_{n+1}$, hence

$$\begin{aligned} F(n+1) - F(n) &= (p_{n+1} - p_n) \cdot \frac{f(\theta_n)}{\theta_n} \phi(\theta_n) \sim k \frac{(p_{n+1} - p_n)f(p_n)}{p_n} \\ &= k \frac{p_{n+1} - p_n}{\log p_n} \cdot \frac{F(n) \log p_n}{p_n} \sim k \cdot \frac{(p_{n+1} - p_n)}{\log p_n} \cdot \frac{F(n)}{n}. \end{aligned}$$

Using Lemma 3, it follows that

$$(5) \quad \frac{p_{n+1} - p_n}{\log p_n} \sim 1.$$

The same conclusion (5) is implied by the hypothesis of concavity of the sequence $(F(n))_{n \geq 1}$. In both cases we obtain (5), which manifestly contradicts (2) and (3).

Remark. The function $f(x) = x^k$ with $k \neq 0$, $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomial functions of different degrees, as well as $f(x) = \frac{x}{\log x}$ are examples of functions for which the sequence $(f(p_n))_{n \geq 1}$ is neither convex nor concave.

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References

- [1] P. ERDŐS and P. TURÁN, On some new question on the distribution of prime numbers, *Bulletin Amer. Math. Soc.* Vol. **54** (1948), 371–378.
- [2] C. POMERANCE, The prime number graph, *Math. Comp.* **33** (1979), 399–408.
- [3] P. ERDŐS, Some problems on consecutive prime numbers, *Mathematica* **19** (1972), 91–95.
- [4] P. ERDŐS and H. DAVENPORT, Small differences between prime numbers, *Proc. Royal Soc. Ser. A*, **253** (1966), 1–18.
- [5] P. ERDŐS, Problems and results on the difference of the consecutive primes, *Publ. Math. Debrecen* **1** (1949), 3–37.

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