

Yau's problem on Einstein field equation

By SHARIEF DESHMUKH (Riyadh)

Abstract. In this short note it is shown that on an n -dimensional compact connected positively curved Riemannian manifold (M, g) without boundary, a symmetric tensor field $T(X, Y) = g(A(X), Y)$ satisfies the Einstein field equation $R_{ij} - \frac{S}{2}g_{ij} = T_{ij}$ if and only if the following conditions are satisfied:

- (i) $\text{tr}.A = -\frac{(n-2)}{2}S$,
- (ii) $(\nabla A)(X, Y) - (\nabla A)(Y, X) = \frac{1}{2}R_0(X, Y)\text{grad } S - F(X, Y)$,
 where R_{ij} is the Ricci tensor, S the scalar curvature, F the divergence of the curvature tensor field and $R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y$.

1. Introduction

In a problem suggested by Yau, it is required to find necessary and sufficient conditions on a symmetric tensor T_{ij} on a compact manifold so that one can find a metric g_{ij} to satisfy the Einstein field equation

$$R_{ij} - \frac{S}{2}g_{ij} = T_{ij},$$

where R_{ij} is the Ricci tensor and S is the scalar curvature (cf. [3], Problem 20, p. 675). Let (M, g) be an n -dimensional compact Riemannian manifold with covariant derivative operator ∇ with respect to the Riemannian connection. Then the divergence of the curvature tensor field R is a tensor field F of type $(1, 2)$ defined by

$$F(X, Y) = \sum_i (\nabla_{e_i} R)(X, Y)e_i, \quad X, Y \in \mathfrak{X}(M),$$

Mathematics Subject Classification: 83C05.

Key words and phrases: curvature tensor field, Ricci tensor, scalar curvature, divergence.

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M . For a symmetric tensor field T of type $(0, 2)$, there is an associated tensor field A of type $(1, 1)$ given by $T(X, Y) = g(AX, Y)$, $X, Y \in \mathfrak{X}(M)$, and its covariant derivative is given by

$$(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y), \quad X, Y \in \mathfrak{X}(M).$$

We also consider a tensor field R_0 of type $(1, 3)$ defined by

$$R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad X, Y, Z \in \mathfrak{X}(M).$$

In this paper we prove the following

Theorem. *Let (M, g) be an n -dimensional compact and connected positively curved Riemannian manifold without boundary. Then a symmetric tensor field $T(X, Y) = g(AX, Y)$ on M satisfies the Einstein field equation*

$$R_{ij} - \frac{S}{2}g_{ij} = T_{ij}$$

if and only if the following conditions are satisfied:

- (i) $tr.A = -\frac{(n-2)}{2}S$
- (ii) $(\nabla A)(X, Y) - (\nabla A)(Y, X) = \frac{1}{2}R_0(X, Y)\text{grad } S - F(X, Y)$, where R_{ij} is the Ricci tensor, S the scalar curvature, F the divergence of the curvature tensor field and $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M .

This theorem can be considered as a result in the direction of the above mentioned problem of YAU [3].

2. Preliminaries

Let (M, g) be an n -dimensional Riemannian manifold and Ric be the Ricci tensor field of M . The Ricci operator $Q : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is defined by

$$\text{Ric}(X, Y) = g(Q(X), Y), \quad X, Y \in \mathfrak{X}(M),$$

and the scalar curvature S of M is given by

$$S = \sum_i \text{Ric}(e_i, e_i),$$

where $\{e_i, \dots, e_n\}$ is a local orthonormal frame on M .

We have the following well known (cf. [2])

Lemma 2.1. $\frac{1}{2} \text{grad } S = \sum_i (\nabla Q)(e_i, e_i).$

The divergence of the curvature tensor field R is a tensor field F given by

$$(2.1) \quad F(X, Y) = \sum_i (\nabla_{e_i} R)(X, Y)e_i, \quad X, Y \in \mathfrak{X}(M)$$

Using (2.1), the second Bianchi identity, and the following expression for the Ricci operator Q ,

$$Q(X) = \sum_i R(X, e_i)e_i, \quad X \in \mathfrak{X}(M),$$

the following lemma can be easily proved.

Lemma 2.2. $(\nabla Q)(X, Y) - (\nabla Q)(Y, X) = -F(X, Y), \quad X, Y \in \mathfrak{X}(M).$

For a symmetric tensor field T of type $(0, 2)$ on M , we define a tensor field A of type $(1, 1)$ by

$$T(X, Y) = g(AX, Y), \quad X, Y \in \mathfrak{X}(M),$$

which is also symmetric. We also define a curvature-like tensor field R_0 by

$$(2.2) \quad R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y$$

and a tensor field B of type $(1, 1)$ by

$$(2.3) \quad B = A - Q.$$

The following lemma is a direct consequence of Lemma 2.2 and equation (2.3).

Lemma 2.3. *If $(\nabla A)(X, Y) - (\nabla A)(Y, X) = \frac{1}{2}R_0(X, Y) \text{grad } S - F(X, Y)$, then the following hold:*

- (i) $(\nabla B)(X, Y) - (\nabla B)(Y, X) = \frac{1}{2}R_0(X, Y) \text{grad } S$
- (ii) $(\nabla^2 B)(X, Y, Z) - (\nabla^2 B)(X, Z, Y) = \frac{1}{2}R_0(X, Y)\nabla_X \text{grad } S.$

Lemma 2.4. *Let A be a symmetric tensor field on an n -dimensional connected Riemannian manifold (M, g) satisfying*

$$(i) \quad tr.A = -\frac{(n-2)}{2}S$$

$$(ii) \quad (\nabla A)(X, Y) - (\nabla A)(Y, X) = \frac{1}{2}R_0(X, Y) \text{grad } S - F(X, Y).$$

Then $\sum_i (\nabla A)(e_i, e_i) = 0$ for a local orthonormal frame $\{e_1, \dots, e_n\}$.

PROOF. For $X \in \mathfrak{X}(M)$, as $tr.A = -\frac{(n-2)}{2}S$, we get

$$\sum_i g((\nabla A)(X, e_i), e_i) = -\frac{(n-2)}{2}g(\text{grad } S, X).$$

Now, using condition (ii) of the statement, we arrive at

$$\begin{aligned} \sum_i g((\nabla A)(e_i, X), e_i) + \frac{1}{2} \sum_i g(R_0(X, e_i) \text{grad } S, e_i) - \sum_i g(F(X, e_i), e_i) \\ = -\frac{(n-2)}{2}g(\text{grad } S, X). \end{aligned}$$

Using (2.2) in the above equation we arrive at

$$(2.4) \quad \sum_i g((\nabla A)(e_i, X), e_i) - \sum_i g(F(X, e_i), e_i) = \frac{1}{2}g(\text{grad } S, X).$$

Now equation (2.1) gives

$$\begin{aligned} \sum_i g(F(X, e_i), e_i) &= \sum_{ik} (\nabla_{e_k} R)(X, e_i; e_k, e_i) = -\sum_k (\nabla_{e_k} \text{Ric})(X, e_k) \\ &= -\sum_k g(X, (\nabla Q)(e_k, e_k)) = -\frac{1}{2}g(X, \text{grad } S). \end{aligned}$$

Thus the equation (2.4) gives $\sum_i g((\nabla A)(e_i, X), e_i) = 0$, or $g(X, \sum_i (\nabla A)(e_i, e_i)) = 0$, which proves the lemma. $\square$

Finally, we prove the following lemma which is the main ingredient in the proof of the main theorem.

Lemma 2.5. *Let A be a symmetric tensor field on an n -dimensional connected Riemannian manifold (M, g) satisfying*

$$(i) \quad tr.A = -\frac{(n-2)}{2}S,$$

$$(ii) \quad (\nabla A)(X, Y) - (\nabla A)(Y, X) = \frac{1}{2}R_0(X, Y) \text{grad } S - F(X, Y).$$

Then $B = A - Q$ satisfies $\|\nabla B\|^2 \geq \frac{n}{4} \|\text{grad } S\|^2$, and for positively curved M the equality holds if and only if $B = -\frac{S}{2}I$.

PROOF. From Lemmas 2.1 and 2.4 we have

$$(2.5) \quad \sum_i (\nabla B)(e_i, e_i) = -\frac{1}{2} \text{grad } S.$$

Define $C : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by $C(X) = B(X) + \frac{S}{2}X$, $X \in \mathfrak{X}(M)$. Then we have

$$(\nabla C)(X, Y) = (\nabla B)(X, Y) + \frac{1}{2}g(\text{grad } S, X)Y,$$

and consequently

$$(2.6) \quad \begin{aligned} \|\nabla C\|^2 &= \|\nabla B\|^2 + \frac{n}{4} \|\text{grad } S\|^2 \\ &\quad + \sum_{ij} g((\nabla B)(e_i, e_j), g(\text{grad } S, e_i)e_j). \end{aligned}$$

Note that B is symmetric as both A and Q are symmetric, and therefore we have

$$g((\nabla B)(X, Y), Z) = g(Y, (\nabla B)(X, Z)), \quad X, Y, Z \in \mathfrak{X}(M).$$

Using this equation and Lemma 2.3, we compute

$$\begin{aligned} &\sum_{ij} g((\nabla B)(e_i, e_j), g(\text{grad } S, e_i)e_j) \\ &= \sum_{ij} g(\text{grad } S, e_i)g\left(B(e_j, e_i) + \frac{1}{2}R_0(e_i, e_j)\text{grad } S, e_j\right) \\ &= \sum_j g(\text{grad } S, (\nabla B)(e_j, e_j)) + \frac{1}{2} \sum_j g(R_0(\text{grad } S, e_j)\text{grad } S, e_j). \end{aligned}$$

Now use this equation, (2.2), and (2.5) in (2.6) to arrive at

$$\|\nabla C\|^2 = \|\nabla B\|^2 - \frac{n}{4} \|\text{grad } S\|^2,$$

which proves the inequality $\|\nabla B\|^2 \geq \frac{n}{4} \|\text{grad } S\|^2$.

Next suppose that M is positively curved and the equality holds. Then we shall have $\nabla C = 0$, which gives $C = \lambda I$ (as M being positively

curved, it is irreducible) for a constant λ . Thus we have $n\lambda = \text{tr}.B + \frac{nS}{2} = \text{tr}.A - \text{tr}.Q + \frac{nS}{2} = 0$, where we have used condition (i); consequently $\lambda = 0$ and this proves $B = -\frac{S}{2}I$. $\square$

3. Proof of the Theorem

If the tensor field T satisfies the Einstein field equation, then we have $A = Q - \frac{S}{2}I$, and from Lemma 2.2 we get the conditions (i), (ii). Conversely suppose that the given conditions are satisfied. Define $f : M \rightarrow R$ by $f = \frac{1}{2}\|B\|^2$. Then choosing a local orthonormal frame $\{e_1, \dots, e_n\}$ on M , we compute the Hessian H_f of f and obtain

$$H_f(X, X) = \sum_{ij} g((\nabla B)(X, e_i), e_j)^2 + \sum_i g((\nabla^2 B)(X, X, e_i), B(e_i)).$$

Thus the Laplacian $\Delta f = \sum_k H_f(e_k, e_k)$ is given by

$$(3.1) \quad \Delta f = \|\nabla B\|^2 + \sum_{ik} g((\nabla^2 B)(e_k, e_k, e_i), B(e_i)).$$

The equation (2.5) gives

$$(3.2) \quad \sum_k (\nabla^2 B)(e_i, e_k, e_k) = -\frac{1}{2} \nabla_{e_i} \text{grad } S,$$

and the Ricci identity implies

$$(3.3) \quad (\nabla^2 B)(e_k, e_i, e_k) = (\nabla^2 B)(e_i, e_k, e_k) + R(e_k, e_i)Be_k - BR(e_k, e_i)e_k.$$

Thus, using (2.2), (3.2), (3.3) and Lemma 2.3 in (3.1), we arrive at

$$\begin{aligned} \Delta f &= \|\nabla B\|^2 - \frac{1}{2} \sum_i g(\nabla_{e_i} \text{grad } S, B(e_i)) + \sum_{ik} [R(e_k, e_i; Be_k, Be_i) \\ &\quad - R(e_k, e_i; e_k, B^2 e_i)] + \frac{1}{2} \sum_{ik} g(\nabla_{e_k} \text{grad } S, e_i) g(e_k, Be_i) \\ (3.4) \quad &\quad - \frac{1}{2} \sum_{ik} g(\nabla_{e_k} \text{grad } S, e_k) g(e_i, Be_i) \\ &= \|\nabla B\|^2 + \frac{n}{4} S \Delta S + \sum_{ik} [R(e_k, e_i; Be_k, Be_i) - R(e_k, e_i; e_k, B^2 e_i)], \end{aligned}$$

where we have used $\text{tr} B = -\frac{n}{2}S$, which follows from condition (i) and $B = A - Q$. Next, we choose a local orthonormal frame $\{e_1, \dots, e_n\}$ which diagonalizes B with $B(e_i) = \mu_i e_i$, and compute

$$\begin{aligned}
 (3.5) \quad & \sum_{ik} [R(e_k, e_i; B e_k, B e_i) - R(e_k, e_i; e_k, B^2 e_i)] \\
 &= \sum_{ik} \mu_i^2 K_{ik} - \mu_i \mu_k K_{ik} = \frac{1}{2} \sum_{ik} 2\mu_i^2 K_{ik} - 2\mu_i \mu_k K_{ik}, \\
 &= \frac{1}{2} \left[\sum_{ik} \mu_i^2 K_{ik} + \mu_k^2 K_{ik} - 2\mu_i \mu_k K_{ik} \right] = \frac{1}{2} \sum_{ik} (\mu_i - \mu_k)^2 K_{ik},
 \end{aligned}$$

where $K_{ik} = R(e_k, e_i; e_i, e_k)$ is the sectional curvature of the plane section spanned by $\{e_i, e_k\}$. Using (3.5) in (3.4) and integrating the resulting equation we get

$$\int_M \left\{ \|\nabla B\|^2 + \frac{n}{4} S \Delta S + \frac{1}{2} \sum_{ik} (\mu_i - \mu_k)^2 K_{ik} \right\} dv = 0.$$

Integrating by parts the second term in the above integral, we arrive at

$$\int_M \left\{ \|\nabla B\|^2 - \frac{n}{4} \|\text{grad } S\|^2 \right\} dv + \frac{1}{2} \int_M \left\{ \sum_{ik} (\mu_i - \mu_k)^2 K_{ik} \right\} dv = 0.$$

Since $K_{ik} > 0$, the above integral together with Lemma 2.5 gives

$$\|\nabla B\|^2 = \frac{n}{4} \|\text{grad } S\|^2,$$

and this equality, again by Lemma 2.5, implies $B = -\frac{S}{2}I$ and consequently the Einstein equation $A = Q - \frac{S}{2}I$.

References

- [1] I. CHAVEL, Riemannian Geometry: A modern introduction, *Cambridge Univ. Press*, no. 108, 1993.
- [2] B. O'NEILL, Semi-Riemannian geometry, *Academic Press*, 1983.

- [3] S. T. YAU, Problem section, Seminar on Differential Geometry, *Princeton Univ. Press*, 1982.

SHARIEF DESHMUKH
DEPARTMENT OF MATHEMATICS
KING SAUD UNIVERSITY
P.O. BOX 2455 RIYADH 11451
SAUDI ARABIA

E-mail: shariefd@ksu.edu.sa

(Received December 12, 1998; file received September 29, 1999)