

## On Laplace and Fourier transforms of periodically monotone functions and distributions

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**Abstract.** In this paper we introduce the periodically monotone functions and find their Laplace transform and Fourier transform (in distribution sense). Finally we introduce the periodically monotone distributions and find their Fourier transform.

### Introduction

In [1] we introduced the periodically monotone sequences in  $\mathbb{R}$  and studied some properties of them.

In the first section of this paper we extend this notion by introducing the periodically monotone sequences in  $\mathbb{C}$  and the periodically monotone functions.

In the second section we find the Laplace transform of these sequences and functions, obtaining an easy generalization of the well known formula that gives the Laplace transform of a periodic function (see [2] p. 109 for instance).

In the third section we introduce the notion of a distribution generated by a sequence and, after having proved that the periodically monotone functions are slowly increasing (Lemma 3.1), we find the Fourier transform of the tempered distributions generated by them (Theorem 3.1).

Finally we introduce the notion of periodically monotone distributions and find their Fourier transform (Theorem 3.2).

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The results of this reaserch can have many applications; for instance, in electronic engineering the periodically monotone functions can be employed as test signals to characterize the behaviour of non linear control systems.

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### 1. Periodically monotone sequences and functions

Let us denote by  $I$  one of the sets  $\mathbb{N}$ ,  $\mathbb{Z}$  and let  $\{a_n\}_{n \in I}$  be a sequence in  $\mathbb{C}$ .

*Definition 1.1.* The sequence  $\{a_n\}_{n \in I}$  is called periodic if there exists a natural number  $q$  such that

$$(1) \quad a_{n+q} = a_n \quad \forall n \in I;$$

A natural number  $q$ , for which (1) holds, is called period.

*Definition 1.2.* The sequence  $\{a_n\}_{n \in I}$  is called periodically monotone if there exist a natural number  $q$  and a periodic sequence  $\{k_n\}_{n \in I}$  in  $\mathbb{C}$  of period  $p$  ( $p$  is a divisor of  $q$ ) such that<sup>1</sup>

$$(2) \quad a_{n+q} = a_n + k_n, \quad \forall n \in I.$$

In particular, if  $a_n \in \mathbb{R}$ ,  $\forall n \in I$  and if  $k_n = k \in \mathbb{R}^+$  ( $k \in \mathbb{R}^-$ )  $\forall n \in I$ , the sequence  $\{a_n\}_{n \in I}$  is called periodically increasing (decreasing). A natural number  $q$ , for which (2) holds, is called period; the sequence  $\{k_n\}_{n \in I}$  (the number  $k$ ) is called monotony sequence (constant).

Let us denote by  $D$  one of the sets  $\mathbb{R}$  or  $[0, +\infty[$ .

*Definition 1.3.* A function  $f : D \rightarrow \mathbb{C}$  is called periodically monotone if there exist a number  $q \in \mathbb{R}^+$  and a function  $g : D \rightarrow \mathbb{C}$  that can be periodic with period  $p$  (here  $q$  is a multiple of  $p$ ) or constant, such that<sup>2</sup>

$$(3) \quad f(t+q) = f(t) + g(t), \quad \forall t \in D.$$

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<sup>1</sup>If  $k_n = 0$ ,  $\forall n \in I$  then  $\{a_n\}_{n \in I}$  is periodic; if  $k_n = k$ ,  $\forall n \in I$  ( $k \neq 0$ ) and  $q = 1$  then  $\{a_n\}_{n \in I}$  is an arithmetical progression.

<sup>2</sup>If  $g(t) = 0$ ,  $\forall t \in D$  then  $f(t)$  is periodic.

In particular, if  $f$  is real valued and if  $g(t) = k \in \mathbb{R}^+$  ( $k \in \mathbb{R}^-$ )  $\forall t \in D$  then  $f$  is called periodically increasing (decreasing). A number  $q \in \mathbb{R}^+$ , for which (3) holds, is called period; the function  $g(t)$  (the number  $k$ ) is called monotony function (constant).

*Definition 1.4.* The sequence  $\{a_n\}_{n \in \mathbb{N}}$  is called  $\mathcal{L}$ -transformable (absolutely  $\mathcal{L}$ -transformable) if the function  $a : [0, +\infty[ \rightarrow \mathbb{C}$ , defined by

$$a(t) = a_n \quad \forall t \in [n-1, n[, \quad \forall n \in \mathbb{N},$$

is  $\mathcal{L}$ -transformable (absolutely  $\mathcal{L}$ -transformable). If  $\{a_n\}_{n \in \mathbb{N}}$  is  $\mathcal{L}$ -transformable we call Laplace transform of  $\{a_n\}_{n \in \mathbb{N}}$ , and denote it by  $\mathcal{L}(\{a_n\}_{n \in \mathbb{N}}, s)$ , the function  $\mathcal{L}(a(t), s)$ .

*Remark 1.1.* If  $\{a_n\}_{n \in \mathbb{N}}$  is absolutely  $\mathcal{L}$ -transformable we have obviously:

$$\begin{aligned} \mathcal{L}(\{a_n\}_{n \in \mathbb{N}}, s) &= \int_0^{+\infty} e^{-st} a(t) dt \\ (4) \quad &= \sum_{n=1}^{+\infty} \int_{n-1}^n e^{-st} a_n dt = \frac{e^s - 1}{s} \sum_{n=1}^{+\infty} a_n e^{-sn}. \end{aligned}$$

Moreover  $\{a_n\}_{n \in \mathbb{N}}$  is absolutely  $\mathcal{L}$ -transformable if and only if the series

$$\sum_{n=1}^{+\infty} a_n e^{-sn}$$

is absolutely convergent.

*Remark 1.2.* If  $\{a_n\}_{n \in \mathbb{N}}$  is periodically monotone with period  $q$  and monotony constant  $k$ , then the function  $a(t)$ , obtained by  $\{a_n\}_{n \in \mathbb{N}}$  accordingly the Definition 1.4, is periodically monotone with period  $q$  and monotony constant  $k$ .

## 2. Laplace transform of the periodically monotone functions

We have the following

**Lemma 2.1.** *Let  $\varphi : [0, +\infty[ \rightarrow \mathbb{C}$  be a periodic function with period  $p$  and (Lebesgue) integrable in  $[0, p]$  or a constant function; moreover let  $q$  a multiple of  $p$  or an arbitrary number in  $\mathbb{R}^+$  if  $\varphi$  is constant and  $\{\alpha_h\}_{h \in \mathbb{N}_0}$  a sequence in  $\mathbb{C}$  such that  $\alpha_h \neq 0$  definitively and  $\lim_{h \rightarrow \infty} |\frac{\alpha_{h+1}}{\alpha_h}| \leq 1$ . Then the function  $\psi : [0, +\infty[ \rightarrow \mathbb{C}$  defined by setting*

$$\psi(t) = \alpha_h \varphi(t), \quad t \in [hq, (h+1)q[ \quad \forall h \in \mathbb{N}_0$$

is absolutely  $\mathcal{L}$ -transformable for  $\operatorname{Re} s > 0$  and we have:

$$(5) \quad \mathcal{L}(\psi(t), s) = \left( \sum_{h=0}^{+\infty} \alpha_h e^{-sqh} \right) \int_0^q e^{-st} \varphi(t) dt.$$

PROOF. To prove that  $\psi(t)$  is absolutely  $\mathcal{L}$ -transformable for  $\operatorname{Re} s > 0$  it is sufficient to prove that the limit

$$\lim_{n \rightarrow \infty} \int_0^{nq} |e^{-st} \psi(t)| dt$$

is finite. Now we have:

$$\begin{aligned} \int_0^{nq} |e^{-st} \psi(t)| dt &= \sum_{h=0}^{n-1} \int_{hq}^{(h+1)q} e^{-\operatorname{Re} st} |\alpha_h| |\varphi(t)| dt \\ &= \sum_{h=0}^{n-1} |\alpha_h| \int_0^q e^{-\operatorname{Re} s(\tau+hq)} |\varphi(\tau+hq)| d\tau \\ &= \left( \sum_{h=0}^{n-1} |\alpha_h| e^{-\operatorname{Re} sqh} \right) \int_0^q e^{-\operatorname{Re} s\tau} |\varphi(\tau)| d\tau. \end{aligned}$$

Therefore, by observing that the series  $\sum_{h=0}^{+\infty} |\alpha_h| e^{-\operatorname{Re} sqh}$  is convergent for the ratio test, from the previous relation it follows that the limit above is finite.

Now we calculate  $\mathcal{L}(\psi(t), s)$ . Proceeding as before we find the relation

$$\int_0^{nq} e^{-st} \psi(t) dt = \left( \sum_{h=0}^{n-1} \alpha_h e^{-sqh} \right) \int_0^q e^{-s\tau} \varphi(\tau) d\tau.$$

From this, for  $n \rightarrow +\infty$  the thesis follows.  $\square$

**Theorem 2.1.** *Let  $f : [0, +\infty[ \rightarrow \mathbb{C}$  be periodically monotone with period  $q$  and monotony function  $g(t)$ ; moreover let  $f(t)$  and  $g(t)$  be integrable in  $[0, q]$ . Then  $f(t)$  is absolutely  $\mathcal{L}$ -transformable for  $\operatorname{Re} s > 0$  and we have:*

$$(6) \quad \mathcal{L}(f(t), s) = \frac{1}{1 - e^{-sq}} \int_0^q e^{-s\tau} f(\tau) d\tau + \frac{e^{-sq}}{(1 - e^{-sq})^2} \int_0^q e^{-s\tau} g(\tau) d\tau$$

PROOF. To prove that  $f(t)$  is absolutely  $\mathcal{L}$ -transformable for  $\operatorname{Re} s > 0$  it is useful to observe that

$$(7) \quad f(t) = f^*(t) + hg(t), \quad \forall t \in [hq, (h+1)q[, \quad \forall h \in \mathbb{N}_0,$$

where  $f^*(t)$  is the periodic extention to  $[0, +\infty[$  of the restriction of  $f(t)$  to  $[0, q]$ . In fact from 7 and from Lemma 2.1 the thesis follows easily.  $\square$

*Remark 2.1.* Let us observe that if  $f(t)$  is periodically monotone with monotony constant  $k$  then the formula (6) becomes

$$(8) \quad \mathcal{L}(f(t), s) = \frac{1}{1 - e^{-sq}} \left( \int_0^q e^{-s\tau} f(\tau) d\tau + \frac{ke^{-sq}}{s} \right).$$

**Corollary 2.1.** *Let  $\{a_n\}_{n \in \mathbb{N}}$  be periodically monotone with period  $q$  and monotony constant  $k$ . Then  $\{a_n\}_{n \in \mathbb{N}}$  is absolutely  $\mathcal{L}$ -transformable for  $\operatorname{Re} s > 0$  and we have:*

$$(9) \quad \mathcal{L}(\{a_n\}_{n \in \mathbb{N}}, s) = \frac{1}{1 - e^{-sq}} \left( \frac{e^s - 1}{s} \sum_{j=1}^q a_j e^{-sj} + \frac{ke^{-sq}}{s} \right).$$

PROOF. The proof follows easily from Remark 1.2 and from (8).  $\square$

### 3. Fourier transform of periodically monotone functions and distributions

In this section we denote by  $\chi_{[a,b]}$  the characteristic function of the interval  $[a, b]$ , by  $\mathcal{F}(F)$  the Fourier transform of the distribution  $F$ , by  $F_h$  the translated distribution of the distribution  $F$  and by  $S(\mathbb{R})$  the space of  $C^\infty$  functions  $\varphi$  defined in  $\mathbb{R}$ , fastly decreasing at infinity.

*Definition 3.1.* Let  $\{a_n\}_{n \in \mathbb{Z}}$  be a sequence in  $\mathbb{C}$  and let  $a(t)$  be the complex valued function defined by

$$(10) \quad a(t) = a_n, \quad \forall t \in [n-1, n[, \quad \forall n \in \mathbb{Z}.$$

The distribution function generated by  $a(t)$  is called distribution generated by  $\{a_n\}_{n \in \mathbb{Z}}$  and in the sequel we denote it by  $A$ .

We have the following

**Lemma 3.1.** *Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a periodically monotone function with period  $q$  and monotony function  $g(t)$ ; moreover let  $f(t)$  and  $g(t)$  be bounded in  $[0, q]$ . Then  $f(t)$  is slowly increasing.*

PROOF.  $\forall n \in \mathbb{N}$  and  $\forall t \in [(n-1)q, nq[$  we have:

$$(11) \quad \begin{aligned} |f(t)| &\leq \sup_{[0,q]} |f(\tau)| + (n-1) \sup_{[0,q]} |g(\tau)| \\ &\leq \max \left( \sup_{[0,q]} |f(\tau)|, \frac{1}{q} \sup_{[0,q]} |g(\tau)| \right) (1 + (n-1)q) \\ &\leq \max \left( \sup_{[0,q]} |f(\tau)|, \frac{1}{q} \sup_{[0,q]} |g(\tau)| \right) (1 + t). \end{aligned}$$

Similarly,  $\forall n \in \mathbb{Z}$ ,  $n \leq 0$  and  $\forall t \in [(n-1)q, nq[$  we have:

$$(12) \quad \begin{aligned} |f(t)| &\leq \sup_{[0,q]} |f(\tau)| + |n-1| \sup_{[0,q]} |g(\tau)| \\ &= \sup_{[0,q]} |f(\tau)| + \sup_{[0,q]} |g(\tau)| + |n| \sup_{[0,q]} |g(\tau)| \\ &\leq \max \left( \sup_{[0,q]} |f(\tau)| + \sup_{[0,q]} |g(\tau)|, \frac{1}{q} \sup_{[0,q]} |g(\tau)| \right) (1 + |n|q) \\ &\leq \max \left( \sup_{[0,q]} |f(\tau)| + \sup_{[0,q]} |g(\tau)|, \frac{1}{q} \sup_{[0,q]} |g(\tau)| \right) (1 + |t|). \end{aligned}$$

From (11) and (12) the thesis follows.  $\square$

*Remark 3.1.* If  $\{a_n\}_{n \in \mathbb{Z}}$  is periodically monotone, then the distribution  $A$  generated by  $\{a_n\}_{n \in \mathbb{Z}}$  is a tempered distribution.

**Theorem 3.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a periodically monotone function with period  $q$  and monotony function  $g(t)$ ; moreover let  $f(t)$  and  $g(t)$  be measurable and bounded in  $[0, q[$ . Then the Fourier transform of the tempered distribution  $F$  generated by the function  $f(t)$  is given by the formula<sup>3</sup>

$$(13) \quad \begin{aligned} \mathcal{F}(F) = & \sum_{h=-\infty}^{+\infty} e^{-2\pi i t q h} \int_0^q f(\tau) e^{-2\pi i \tau t} d\tau \\ & + \sum_{h=-\infty}^{+\infty} h e^{-2\pi i t q h} \int_0^q g(\tau) e^{-2\pi i \tau t} d\tau. \end{aligned}$$

PROOF.  $\forall \varphi \in S(\mathbb{R})$  we have

$$(14) \quad \begin{aligned} \langle F, \varphi \rangle &= \int_{-\infty}^{+\infty} f(t) \varphi(t) dt = \sum_{h=-\infty}^{+\infty} \int_{hq}^{(h+1)q} f(t) \varphi(t) dt \\ &= \sum_{h=-\infty}^{+\infty} \int_0^q f(\tau + hq) \varphi(\tau + hq) d\tau \\ &= \sum_{h=-\infty}^{+\infty} \int_0^q (f(\tau) + hg(\tau)) \varphi(\tau + hq) d\tau \\ &= \sum_{h=-\infty}^{+\infty} \left( \int_0^q f(\tau) \varphi(\tau + hq) d\tau + h \int_0^q g(\tau) \varphi(\tau + hq) d\tau \right) \\ &= \sum_{h=-\infty}^{+\infty} \int_0^q f(\tau) \varphi(\tau + hq) d\tau + \sum_{h=-\infty}^{+\infty} h \int_0^q g(\tau) \varphi(\tau + hq) d\tau. \end{aligned}$$

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<sup>3</sup>Observe that the series which appear in (13) are convergent in the sense of weak topology of  $S'(\mathbb{R})$ .

The relation (14) implies that

$$(15) \quad F = \sum_{h=-\infty}^{+\infty} (\chi_{[0,q]}[f(t)])_{hq} + \sum_{h=-\infty}^{+\infty} h (\chi_{[0,q]}[g(t)])_{hq}.$$

From this relation we obtain

$$\mathcal{F}(F) = \sum_{h=-\infty}^{+\infty} e^{-2\pi i t q h} \mathcal{F}(\chi_{[0,q]}[f(t)]) + \sum_{h=-\infty}^{+\infty} h e^{-2\pi i t q h} \mathcal{F}(\chi_{[0,q]}[g(t)]),$$

and finally from this relation, by using the Proposition 29.1 of p. 307 of [3], the thesis follows.  $\square$

*Remark 3.2.* Let us observe that if  $f$  is periodically monotone with period  $q$  and monotony constant  $k$  then the formula (13) becomes

$$(16) \quad \begin{aligned} \mathcal{F}(F) = & \sum_{h=-\infty}^{+\infty} e^{-2\pi i t q h} \int_0^q f(\tau) e^{-2\pi i \tau t} d\tau \\ & + k \sum_{h=-\infty}^{+\infty} h e^{-2\pi i t q h} \frac{1 - e^{-2\pi i q t}}{2\pi i t}. \end{aligned}$$

**Corollary 3.1.** Let  $\{a_n\}_{n \in \mathbb{Z}}$  be periodically monotone with period  $q$  and monotony constant  $k$ . Then the Fourier transform of the distribution  $A$  generated by  $\{a_n\}_{n \in \mathbb{Z}}$  is given by the formula

$$(17) \quad \begin{aligned} \mathcal{F}(A) = & \sum_{h=-\infty}^{+\infty} e^{-2\pi i t q h} \sum_{j=1}^q a_j \frac{e^{-2\pi i (j-1)t} - e^{-2\pi i j t}}{2\pi i t} \\ & + k \sum_{h=-\infty}^{+\infty} h e^{-2\pi i t q h} \frac{1 - e^{-2\pi i q t}}{2\pi i t}. \end{aligned}$$

*Remark 3.3.* Observe that the formula (15) holds if we suppose that  $f(t)$  and  $g(t)$  are only integrable in  $[0, q]$  rather than mesurable and bounded (obviously in this case  $F$  may not be a tempered distribution). Therefore the formula (15) suggests to define the notion of periodically monotone distribution in the following way.

*Definition 3.2.* A distribution  $T$  is called periodically monotone with period  $q$  if there exist two distributions  $T_0$  and  $U_0$  with support contained in  $[0, q]$  and such that<sup>4</sup>

$$(18) \quad T = \sum_{h=-\infty}^{+\infty} (T_0)_{hq} + \sum_{h=-\infty}^{+\infty} h (U_0)_{hq}.$$

*Remark 3.4.* Let us observe that the distribution  $\sum_{h=-\infty}^{+\infty} (T_0)_{hq}$  is a periodic distribution with period  $q$ , therefore it is a tempered distribution. Instead the distribution  $\sum_{h=-\infty}^{+\infty} h (U_0)_{hq}$  in general is not a tempered distribution.

Also observe that if  $T$  verifies (18) then we have:

$$T' = \sum_{h=-\infty}^{+\infty} (T'_0)_{hq} + \sum_{h=-\infty}^{+\infty} h (U'_0)_{hq},$$

and thus  $T'$  is a periodically monotone distribution with period  $q$ .

For the periodically monotone tempered distributions the following theorem holds.

**Theorem 3.2.** *Let  $T$  be a periodically monotone tempered distribution verifying (18) in  $S'(\mathbb{R})$ . Then the Fourier transform of  $T$  is given by the formula*

$$(19) \quad \mathcal{F}(T) = \sum_{h=-\infty}^{+\infty} e^{-2\pi i t q h} \langle T_0, e^{-2\pi i \tau t} \rangle + \sum_{h=-\infty}^{+\infty} h e^{-2\pi i t q h} \langle U_0, e^{-2\pi i \tau t} \rangle,$$

where  $\langle T_0, e^{-2\pi i \tau t} \rangle$  and  $\langle U_0, e^{-2\pi i \tau t} \rangle$  are functions of  $t$ .

PROOF. From (18) we have

$$\mathcal{F}(T) = \sum_{h=-\infty}^{+\infty} e^{-2\pi i t q h} \mathcal{F}(T_0) + \sum_{h=-\infty}^{+\infty} h e^{-2\pi i t q h} \mathcal{F}(U_0),$$

and from this relation, by using the Proposition 29.1 of [3] again, the thesis follows.  $\square$

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<sup>4</sup> $T_0$  and  $U_0$  are tempered distributions, for all distributions with compact support are tempered distributions (see [3] p. 274) for instance.

*Example 1.* Let us take the Dirac distribution  $\delta$  defined by the formula

$$\langle \delta, \varphi \rangle = \varphi(0), \quad \forall \varphi \in S(\mathbb{R}).$$

Then the series

$$\sum_{h=-\infty}^{+\infty} \delta_h \quad \text{and} \quad \sum_{h=-\infty}^{+\infty} h\delta_h$$

are convergent in  $S'(\mathbb{R})$ , and therefore the distribution

$$T = \sum_{h=-\infty}^{+\infty} \delta_h + \sum_{h=-\infty}^{+\infty} h\delta_h$$

is a periodically monotone tempered distribution with period 1. Moreover by virtue of Remark 3.4 also  $T^{(k)}$ ,  $\forall k \in \mathbb{N}$  is a periodically monotone tempered distribution with period 1.

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