

Randers spaces with the h -curvature tensor H dependent on position alone

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Abstract. We give an example of Finsler space having the Berwald h -curvature tensor H which is independent of the direction arguments y^i [3].

1. Introduction

In our previous paper [3] we were concerned with various sets of special kinds of Finsler spaces. Among them we pay attention to the two sets

$B(n)$... n -dim. Berwald spaces,

$Hx(n)$... n -dim. spaces having the Berwald h -curvature tensor H
dependent on the position alone.

The inclusion relation $B(n) \subset Hx(n)$ is obvious, but any example of a Finsler space belonging to $Hx(n)$ but not to $B(n)$ has not been given in the paper.

The purpose of the present paper is to give an example of such a Finsler space, a *Randers space* $F^n = (M^n, L = \alpha + \beta)$. Its metric L consists of a Riemannian metric α ($\alpha^2 = a_{ij}(x)y^i y^j$) and a differential one-form $\beta = b_i(x)y^i$.

The Riemannian space $R^n = (M^n, \alpha)$ is said to be associated with F^n . Let $\gamma_j^i(x)$ be the Christoffel symbols of R^n . Then we have the Levi-Civita connection $\gamma = \{\gamma_j^i\}$ in R^n and the induced Finsler connection

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$F\gamma = \{\gamma_j^i k, \gamma_0^i j, 0\}$ in F^n . (Throughout the paper the subscript 0 denotes the transvection by y^i .) The h - and v -covariant differentiations in $F\gamma$ are denoted by $(; , \cdot)$ respectively. Let us use the following symbols:

$$\begin{aligned} r_{ij} &= \frac{1}{2}(b_{i;j} + b_{j;i}), & r_2 &= \frac{1}{2}r_{00}, & s_{ij} &= \frac{1}{2}(b_{i;j} - b_{j;i}), \\ y_i &= a_{ir}y^r, & b^i &= a^{ir}b_r, & s^i_j &= a^{ir}s_{rj}, & s_i &= b_rs^r_i. \end{aligned}$$

We are interested in Randers spaces from the standpoint of not only applications but also pure geometry [1, 1.4]. For instance it is a remarkable result [4], [7] that a Randers space is a *Berwald space*, if and only if $b_{i;j} = 0$. Next a Randers space is a *Douglas space* which has been introduced by the authors [2], if and only if $b_{i;j} - b_{j;i} = 0$, that is, β is a closed form.

We shall adopt here Randers spaces to give an example of Finsler spaces belonging to the set $Hx(n)$. The quantities $G^i(x, y)$ appearing in the equations $d^2x^i/ds^2 + 2G^i(x, dx/ds) = 0$ of geodesic in the Randers space F^n are written as [7]

$$(1.1) \quad 2G^i = \gamma_0^i{}_0 + 2B^i$$

where the tensor $B^i(x, y)$ is of the form

$$(1.2) \quad LB^i = B_3^i + \alpha B_2^i,$$

$$(1.2a) \quad B_2^i = \beta s^i{}_0 - s_0 y^i, \quad B_3^i = \alpha^2 s^i{}_0 + r_2 y^i.$$

In the following the subscripts $a = 2, \dots, 9$ denote that the entity is a homogeneous polynomial in y^i of degree a ; B_a^i , $a = 2, 3$, of (1.2a) are homogeneous polynomials in y^i of degree two and three respectively.

2. The h -curvature tensor of a Randers space

We deal with a Randers space $F^n = (M^n, L = \alpha + \beta)$ equipped with the Berwald connection $B\Gamma = \{G_j^i, G_j^i k, 0\}$, and denote by $H = (H_i^h{}_{jk})$ and $R^1 = (R^h{}_{jk})$ the h -curvature tensor and the $(v)h$ -torsion tensor respectively. Then we have well-known relations

$$(2.1) \quad (i) \quad R^h{}_{jk} = H_0^h{}_{jk}, \quad (ii) \quad H_i^h{}_{jk} = R^h{}_{jk \cdot i}.$$

R^1 is defined by

$$(2.2) \quad R^h_{jk} = \partial_k G^h_j - \partial_j G^h_k - G^h_r G^r_k + G^h_r G^r_j.$$

Further we shall take notice of the relation [5, (18.23)]

$$(2.3) \quad R^h_{jk} = \frac{1}{3}(R^h_{0k \cdot j} - R^h_{0j \cdot k}),$$

for the later use. Then, to find the tensor H , we first construct R^h_{0k} and then use (2.3) and (ii) of (2.1).

It follows from (2.2) that

$$R^h_{0k} = 2\partial_k G^h - y^j \partial_j G^h_k - G^h_r G^r_k + 2G^h_r G^r.$$

Then, using (1.1) and the h -covariant differentiation (;) in $F\gamma$, we obtain

$$(2.4) \quad R^h_{0k} = \overset{a}{R}_0{}^h{}_{0k} + 2B^h{}_{;k} - B^h{}_{k;0} + 2B_k{}^h{}_r B^r - B^h{}_r B^r{}_k,$$

where $\overset{a}{R}$ is the curvature tensor of R^n and we put $B^h{}_k = B^h{}_{\cdot k}$ and $B_k{}^h{}_r = B^h{}_{k \cdot r}$.

In the following we shall quote extensively from the paper [6] the procedure in order to obtain R^h_{0k} of the Randers space belonging to the set $Hx(n)$.

First, from (1.2) we have

$$(2.5) \quad L^2 \alpha B^i{}_j = C_4{}^i{}_j + \alpha C_3{}^i{}_j,$$

$$(2.5a) \quad \begin{cases} C_3{}^i{}_j = \alpha^2 B_2{}^i{}_{\cdot j} + \beta B_3{}^i{}_{\cdot j} - B_3{}^i b_j, \\ C_4{}^i{}_j = \alpha^2 (\beta B_2{}^i{}_{\cdot j} + B_3{}^i{}_{\cdot j}) + B_2{}^i (\beta y_j - \alpha^2 b_j) - B_3{}^i y_j, \end{cases}$$

$$(2.6) \quad L^3 \alpha^3 B_j{}^i{}_k = D_6{}^i{}_{jk} + \alpha D_5{}^i{}_{jk},$$

$$(2.6a) \quad \begin{cases} D_5{}^i{}_{jk} = \alpha^2 (\beta C_3{}^i{}_{j \cdot k} + C_4{}^i{}_{j \cdot k}) - 2\alpha^2 C_3{}^i{}_j b_k - 3C_4{}^i{}_j y_k, \\ D_6{}^i{}_{jk} = \alpha^2 (\alpha^2 C_3{}^i{}_{j \cdot k} + \beta C_4{}^i{}_{j \cdot k}) - 2\alpha^2 C_3{}^i{}_j y_k \\ \quad - C_4{}^i{}_j (\beta y_k + 2\alpha^2 b_k). \end{cases}$$

Then we have

$$(2.7) \quad L^4 \alpha^2 B^i_r B^r_j = E_8^i_j + \alpha E_7^i_j,$$

$$(2.7a) \quad \begin{cases} E_7^i_j = C_4^i_r C_3^r_j + C_3^i_r C_4^r_j, \\ E_8^i_j = C_4^i_r C_4^r_j + \alpha^2 C_3^i_r C_3^r_j, \end{cases}$$

$$(2.8) \quad L^3 \alpha B^i_r B^r = F_7^i + \alpha F_6^i,$$

$$(2.8a) \quad F_6^i = C_3^i_r B_3^r + C_4^i_r B_2^r, \quad F_7^i = C_4^i_r B_3^r + \alpha^2 C_3^i_r B_2^r,$$

$$(2.9) \quad L^4 \alpha^3 B_j^i B^r = G_9^i_j + \alpha G_8^i_j,$$

$$(2.9a) \quad \begin{cases} G_8^i_j = D_5^i_{jr} B_3^r + D_6^i_{jr} B_2^r, \\ G_9^i_j = D_6^i_{jr} B_3^r + \alpha^2 D_5^i_{jr} B_2^r. \end{cases}$$

Next we get

$$(2.10) \quad L^2 B^i_{;j} = H_4^i_j + \alpha H_3^i_j,$$

$$(2.10a) \quad \begin{cases} H_3^i_j = \beta B_2^i_{;j} + B_3^i_{;j} - B_2^i(r_{0j} + s_{0j}), \\ H_4^i_j = \alpha^2 B_2^i_{;j} + \beta B_3^i_{;j} - B_3^i(r_{0j} + s_{0j}), \end{cases}$$

$$(2.11) \quad L^3 \alpha B^i_{j;0} = I_6^i_j + \alpha I_5^i_j,$$

$$(2.11a) \quad \begin{cases} I_5^i_j = \beta C_3^i_{j;0} + C_4^i_{j;0} - 4r_2 C_3^i_j, \\ I_6^i_j = \alpha^2 C_3^i_{j;0} + \beta C_4^i_{j;0} - 4r_2 C_4^i_j. \end{cases}$$

Substituting (2.10), (2.11), (2.9) and (2.7) in (2.4), we get R^h_{0k} in the form

$$(2.12) \quad \begin{aligned} L^4 \alpha^3 (R^h_{0k} - \overset{a}{R}_0^h{}_{0k}) &= 2L^2 \alpha^3 (H_4^h_k + \alpha H_3^h_k) \\ &\quad - L \alpha^2 (I_6^h_k + \alpha I_5^h_k) + 2(G_9^h_k + \alpha G_8^h_k) - \alpha(E_8^h_k + \alpha E_7^h_k). \end{aligned}$$

Since we have $(B^i_r B^r)_{;j} = B_j^i_r B^r + B^i_r B^r_j$, (2.8), (2.9) and (2.7) yield

the relation

$$(2.13) \quad \begin{cases} G_8^i{}_j + E_8^i{}_j = \alpha^2(\beta F_6^i{}_j + F_7^i{}_j) - 3\alpha^2 F_6^i b_j - 4F_7^i y_i, \\ G_9^i{}_j + \alpha^2 E_7^i{}_j = \alpha^2(\alpha^2 F_6^i{}_j + \beta F_7^i{}_j) - 3\alpha^2 F_6^i y_i \\ \quad - F_7^i(\beta y_j + 3\alpha^2 b_j). \end{cases}$$

3. The condition for Randers spaces to belong to $Hx(n)$

It is observed that (2.12) is obtained from (2.1) of the paper [6] by the

$$(3.1) \quad \text{change: } (K, \overset{a}{R}) \longrightarrow (0, \overset{a}{R} - R).$$

If we deal with the Randers space F^n belonging to $Hx(n)$, then (i) of (2.1) gives

$$R^h{}_{0k} - \overset{a}{R}_0^h{}_{0k} = (H_r^h{}_{sk}(x) - \overset{a}{R}_r^h{}_{sk}(x))y^r y^s,$$

homogeneous polynomials in y^i of degree two. Consequently the discussions in §2–4 of [6] can be applied without modification. The conclusion in [6] is that

Lemma 3. *Randers spaces of dimension more than two and of constant curvature K are classified as follows*

- (I) *RCG-space:* $r_{ij} = 2c(a_{ij} - b_i b_j)$, $s_{ij} = 0$, $K + c^2 = 0$,
- (II) *RCT-space:* $r_{ij} = 0$, $s_i = 0$, $c = 0$, $s_{ij;k} = K(a_{ik} b_j - a_{jk} b_i)$.

On these conditions the remarkable form of $\overset{a}{R}$ was given by (5.3) with (5.4) of [6].

By the change (3.1) we then have the conclusion as follows:

A Randers space F^n , $n > 2$, belongs to the set $Hx(n)$, if and only if

- (I) *G-type:* $r_{ij} = s_{ij} = 0$,
- (II) *T-type:* $r_{ij} = 0$, $s_i = 0$, $s_{ij;k} = 0$.

In any case we have $c = 0$ and $r_{ij} = 0$, and hence (5.3) with (5.4) of [6] leads to

$$(3.2) \quad R^h{}_{0k} = \overset{a}{R}_0^h{}_{0k} - (s^r s_r) y^h y_k - 3s^h{}_0 s_{0k} + s^h{}_r s^r{}_0 y_k \\ - \alpha^2 s^h{}_r s^r{}_k + y^h s_{0r} s^r{}_k + \{(s^r s_r) \alpha^2 - s_{0r} s^r{}_0\} \delta^h{}_k.$$

We have $b_{i;j} = 0$ for F^n of G-type, and consequently F^n is a Berwald space where $G_j^i = \gamma_j^i(x)$ and the h -curvature tensor H obviously coincides with the Riemannian $\overset{a}{R}$.

On the other hand, for F^n of T-type we have

$$\begin{aligned} r_{ij} = 0 : \quad b_{i;j} &= s_{ij} \text{ (skew-symmetric),} \\ s_i = 0 : \quad s_i &= b^r b_{r;i} = \frac{1}{2}(a^{rs} b_r b_s)_{;i} = 0. \end{aligned}$$

The former shows that b_i is a Killing vector in R^n , and the latter indicates that the length of b_i is constant in R^n . Therefore b_i is the so-called translation. Further $s_{ij;k} = 0$ together with

$$s_{i;j} = (b^r b_{r;i})_{;j} = b^r_{;j} b_{r;i} + b^r b_{r;i;j} = s^r_j s_{ri} + b^r s_{ri;j},$$

leads to $s^r_j s_{ri} = 0$. Therefore (3.2) is reduced to

$$(3.3) \quad R^h_{0k} = \overset{a}{R}_0^h{}_{0k} - 3s^h{}_0 s_{0k}.$$

Then (3.3) gives

$$R^h_{0k;j} = \overset{a}{R}_j^h{}_{0k} + \overset{a}{R}_0^h{}_{jk} - 3(s^h{}_j s_{0k} + s^h{}_0 s_{jk}),$$

and (2.3) yields

$$R^h_{jk} = \frac{1}{3} \left(\overset{a}{R}_j^h{}_{0k} - \overset{a}{R}_k^h{}_{0j} + 2\overset{a}{R}_0^h{}_{jk} \right) - s^h{}_j s_{0k} + s^h{}_k s_{0j} - 2s^h{}_0 s_{jk}.$$

On account of the well-known identities satisfied by $\overset{a}{R}_{hijk}$ we have

$$\overset{a}{R}_{jh0k} - \overset{a}{R}_{kh0j} = -\overset{a}{R}_{hj0k} - \overset{a}{R}_{kh0j} = \overset{a}{R}_{jk0h} = \overset{a}{R}_{0hjk}.$$

Thus we get

$$R^h_{jk} = \overset{a}{R}_0^h{}_{jk} - s^h{}_j s_{0k} + s^h{}_k s_{0j} - 2s^h{}_0 s_{jk},$$

and finally (ii) of (2.1) gives

$$(3.4) \quad H_i^h{}_{jk} = \overset{a}{R}_i^h{}_{jk} - s^h{}_j s_{ik} + s^h{}_k s_{ij} - 2s^h{}_i s_{jk}, \quad s_{ij} = b_{i;j}.$$

Theorem 1. *A Randers space F^n , $n > 2$, has the h -curvature tensor H of the Berwald connection which depends on the position alone, if and only if (I) $b_{i;j} = 0$, F^n being a Berwald space, or (II) b_i is a translation in the associated Riemannian space, that is, $b_{i;j} + b_{j;i} = 0$ and $b^r b_{r;i} = 0$, and that satisfies $b_{i;j;k} = 0$.*

In the case (II) we have $G^i = \gamma_0^i{}_0/2 + \alpha b^i{}_{;0}$ and the tensor H is written in the form (3.4).

4. The two-dimensional case

A Randers space of dimension two is an exceptional case in [6], because “Lemma 1” (p. 256) needs the restriction $n > 2$. However the condition (II) of Theorem 1 may be applicable to the case $n = 2$. Thus this last section is devoted to the consideration of a two-dimensional Randers space F^2 satisfying

$$(4.1) \quad b_{i;j} + b_{j;i} = 2r_{ij} = 0, \quad s_i = 0, \quad s_{ij;k} = 0.$$

Thus (1.2a) gives

$$\begin{aligned} B_2^i &= \beta s^i{}_0, & B_3^i &= \alpha^2 s^i{}_0, & B_2^i{}_{;j} &= s^i{}_0 s_{0j}, & B_3^i{}_{;j} &= 0, \\ B_2^i{}_{;j} &= s^i{}_0 b_j + \beta s^i{}_j, & B_3^i{}_{;j} &= 2s^i{}_0 y_j + \alpha^2 s^i{}_j. \end{aligned}$$

Then (2.5a) yields

$$C_3^i{}_{;j} = 2\beta(s^i{}_0 y_j + \alpha^2 s^i{}_j), \quad C_4^i{}_{;j} = (\alpha^2 + \beta^2)(s^i{}_0 y_j + \alpha^2 s^i{}_j).$$

Next (2.10a) gives $H_3 = H_4 = 0$ and $\beta_{;0} = 0$ leads to $C_{3;0} = C_{4;0} = 0$. Thus (2.11a) gives $I_5 = I_6 = 0$. Further we have $F_6 = F_7 = 0$ from (2.8a) and hence (2.13) leads to $G_8 = -E_8$ and $G_9 = -\alpha^2 E_7$. (2.7a) gives

$$\begin{aligned} E_8^i{}_{;j} &= \alpha^2(\alpha^4 + 6\alpha^2\beta^2 + \beta^4)s^i{}_0 s_{0j}, \\ E_7^i{}_{;j} &= 4\alpha^2\beta(\alpha^2 + \beta^2)s^i{}_0 s_{0j}. \end{aligned}$$

Therefore (2.12) is written as $L^4\alpha^3(R^h{}_{0k} - \overset{a}{R}_0{}^h{}_{0k}) = -3L^4\alpha^3 s^h{}_0 s_{0k}$ which is nothing but (3.3).

Consequently we have

Theorem 2. *Let F^2 be a Randers space of dimension two. If F^2 satisfies $b_{i;j} + b_{j;i} = 0$, $b^r b_{r;i} = 0$ and $b_{i;j;k} = 0$, then the h -curvature tensor H of the Berwald connection depends on the position alone, written in the form (3.4).*

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