

On elements in algebras having finite number of conjugates

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Dedicated to Professor Kálmán Györy on his 60th birthday

Abstract. Let R be a ring with unity and $U(R)$ its group of units. Let $\Delta U = \{a \in U(R) \mid [U(R) : C_{U(R)}(a)] < \infty\}$ be the FC -radical of $U(R)$ and let $\nabla(R) = \{a \in R \mid [U(R) : C_{U(R)}(a)] < \infty\}$ be the FC -subring of R .

An infinite subgroup H of $U(R)$ is said to be an ω -subgroup if the left annihilator of each nonzero Lie commutator $[x, y]$ in R contains only finite number of elements of the form $1 - h$, where $x, y \in R$ and $h \in H$. In the case when R is an algebra over a field F , and $U(R)$ contains an ω -subgroup, we describe its FC -subalgebra and the FC -radical. This paper is an extension of [1].

1. Introduction

Let R be a ring with unity and $U(R)$ its group of units. Let

$$\Delta U = \{a \in U(R) \mid [U(R) : C_{U(R)}(a)] < \infty\},$$

and

$$\nabla(R) = \{a \in R \mid [U(R) : C_{U(R)}(a)] < \infty\},$$

which are called the FC -radical of $U(R)$ and FC -subring of R , respectively. The FC -subring $\nabla(R)$ is invariant under the automorphisms of R and contains the center of R .

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The investigation of the FC -radical ΔU and the FC -subring $\nabla(R)$ was proposed by S. K. SEHGAL and H. ZASSENHAUS [8]. They described the FC -subring of a $\mathbb{Z}$ -order as a unital ring with a finite $\mathbb{Z}$ -basis and a semisimple quotient ring.

Definition. An infinite subgroup H of $U(R)$ is said to be an ω -subgroup if the left annihilator of each nonzero Lie commutator $[x, y] = xy - yx$ in R contains only a finite number of elements of the form $1 - h$, where $h \in H$ and $x, y \in R$.

The groups of units of the following infinite rings R contain ω -subgroups, of course:

1. Let A be an algebra over an infinite field F . Then the subgroup $U(F)$ is an ω -subgroup.
2. Let $R = KG$ be the group ring of an infinite group G over the ring K . It is well-known (see [6], Lemma 3.1.2, p. 68) that the left annihilator of any $z \in KG$ contains only a finite number of elements of the form $g - 1$, where $g \in G$. Thus G is an ω -subgroup.
3. Let $R = F_\lambda G$ be an infinite twisted group algebra over the field F with an F -basis $\{u_g \mid g \in G\}$. Then the subgroup $\overline{G} = \{\lambda u_g \mid \lambda \in U(F), g \in G\}$ is an ω -subgroup.
4. If A is an algebra over a field F , and A contains a subalgebra D such that $1 \in D$ and D is either an infinite field or a skewfield, then every infinite subgroup of $U(D)$ is an ω -subgroup.

2. Results

In this paper we study the properties of the FC -subring $\nabla(R)$ when R is an algebra over a field F and $U(R)$ contains an ω -subgroup. We show that the set of algebraic elements A of $\nabla(R)$ is a locally finite algebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$ and $A/\mathfrak{J}(A)$ is commutative. As a consequence, we describe the FC -radical ΔU , which is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2. If F is an infinite field then any algebraic unit over F belongs to the centralizer of $\nabla(R)$, and, as a consequence, we obtain that $t(\Delta U)$ is abelian and ΔU is nilpotent of class at most 2. These results are extensions of the results obtained by the author in [1] for groups of units of twisted group algebras.

By the Theorem of B. H. NEUMANN [5], elements of finite order in ΔU form a normal subgroup, which we denote by $t(\Delta U)$, and the factor group $\Delta U/t(\Delta U)$ is a torsion free abelian group. If x is a nilpotent element of the ring R , then the element $y = 1 + x$ is a unit in R , which is called the unipotent element of $U(R)$.

Let $\zeta(G)$ be the center of G and $(g, h) = g^{-1}h^{-1}gh$, where $g, h \in G$.

Lemma 1. *Assume that $U(R)$ has an ω -subgroup. Then all nilpotent elements of the subring $\nabla(R)$ are central in $\nabla(R)$.*

PROOF. Let x be a nilpotent element of $\nabla(R)$. Then $x^k = 0$, and by induction on k we shall prove that $vx = xv$ for all $v \in \nabla(R)$.

Choose an infinite ω -subgroup H of $U(R)$. By Poincaré's Theorem the centralizer S of the subset $\{v, x\}$ in H is a subgroup of finite index in H . Since H is infinite, S is infinite and $fx = xf$ for all $f \in S$. Then xf is nilpotent and $1 + xf$ is a unit in $U(R)$. Since $v \in \nabla(R)$, the set $\{(1 + xf)^{-1}v(1 + xf) \mid f \in S\}$ is finite. Let $v_1, \dots, v_t$ be all the elements of this set and

$$W_i = \{f \in S \mid (1 + xf)^{-1}v(1 + xf) = v_i\}.$$

Obviously, $S = \bigcup W_i$ and there exists an index j such that W_j is infinite. Fix an element $f \in W_j$. Then any element $q \in W_j$ such that $q \neq f$ satisfies

$$(1 + xf)^{-1}v(1 + xf) = (1 + xq)^{-1}v(1 + xq)$$

and

$$v(1 + xf)(1 + xq)^{-1} = (1 + xf)(1 + xq)^{-1}v.$$

Then

$$v\{(1 + xq) + (xf - xq)\}(1 + xq)^{-1} = \{(1 + xq) + (xf - xq)\}(1 + xq)^{-1}v,$$

$$v(1 + x(f - q)(1 + xq)^{-1}) = (1 + x(f - q)(1 + xq)^{-1})v$$

and

$$(1) \quad vx(f - q)(1 + xq)^{-1} = x(f - q)(1 + xq)^{-1}v.$$

Let $xv \neq vx$ and $k = 2$. Then $x^2 = 0$ and $(1 + xq)^{-1} = 1 - xq$. Since f and q belong to the centralizer of the subset $\{x, v\}$, from (1) we have

$$(f - q)vx(1 - xq) = (f - q)x(1 - xq)v,$$

whence $(f-q)(vx-vx^2q-xv+x^2qv) = 0$ and evidently $(f-q)(vx-xv) = 0$. Therefore, $(q^{-1}f-1)(vx-xv) = 0$ for any $q \in W_j$. Since $q^{-1}W_j$ is an infinite subset of the ω -subgroup H , we obtain a contradiction, and thus $vx = xv$.

Let $k > 2$. If $i \geq 1$ then x^{i+1} is nilpotent of index less than k , thus applying an induction on k , first we obtain that $x^{i+1}v = vx^{i+1}$ and then

$$x(f-q)x^i q^i v = (f-q)x^{i+1} q^i v = (f-q)vx^{i+1} q^i = vx(f-q)x^i q^i.$$

Hence

$$\begin{aligned} & vx(f-q)(1-xq+x^2q^2+\cdots+(-1)^{k-1}x^{k-1}q^{k-1}) \\ &= x(f-q)((1-xq)v+(x^2q^2+\cdots+(-1)^{k-1}x^{k-1}q^{k-1})v). \end{aligned}$$

and $(f-q)(vx-xv) = 0$. As before, we have a contradiction in the case $xv \neq vx$.

Thus nilpotent elements of $\nabla(R)$ are central in $\nabla(R)$. $\square$

Lemma 2. *Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω -subgroup. Then the radical $\mathfrak{J}(A)$ of every locally finite subalgebra A of $\nabla(R)$ consists of central nilpotent elements of the subalgebra $\nabla(R)$, and $A/\mathfrak{J}(A)$ is a commutative algebra.*

PROOF. Let $x \in \mathfrak{J}(A)$, then $x \in L$ for some finite dimensional subalgebra L of A . Since L is left Artinian, Proposition 2.5.17 in [7] (p. 185) ensures that $L \cap \mathfrak{J}(A) \subseteq \mathfrak{J}(L)$, moreover $\mathfrak{J}(L)$ is nilpotent. Now $x \in \mathfrak{J}(L)$ implies that x is nilpotent and the application of Lemma 1 gives that x belongs to the center of $\nabla(R)$. Then Theorem 48.3 in [4] (p. 209) will enable us to verify the existence of the decomposition into the direct sum

$$L = Le_1 \oplus \cdots \oplus Le_n \oplus N,$$

where Le_i is a finite dimensional local F -algebra (i.e. $Le_i/\mathfrak{J}(Le_i)$ is a division ring), N is a commutative artinian radical algebra, and $e_1, \dots, e_n$ are pairwise orthogonal idempotents. Since nilpotent elements of $\nabla(R)$ belong to the center of $\nabla(R)$, by Lemma 13.2 of [4] (p. 57) any idempotent e_i is central in L and the subring Le_i of $\nabla(R)$ is also an FC -ring, whence $\mathfrak{J}(Le_i)$ is a central nilpotent ideal.

Suppose that $Le_i/\mathfrak{J}(Le_i)$ is a noncommutative division ring. Then $1 + \mathfrak{J}(Le_i)$ is a central subgroup and

$$U(Le_i)/(1 + \mathfrak{J}(Le_i)) \cong U(Le_i/\mathfrak{J}(Le_i)).$$

Applying HERSTEIN's Theorem [2] we establish that a noncentral unit of $Le_i/\mathfrak{J}(Le_i)$ has an infinite number of conjugates, which is impossible. Therefore, $L/\mathfrak{J}(L)$ is a commutative algebra and from $\mathfrak{J}(L) \subseteq \mathfrak{J}(A)$ and $\mathfrak{J}(L)$ is nil (actually nilpotent) in A , it follows that $A/\mathfrak{J}(A)$ is a commutative algebra. $\square$

Theorem 1. *Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω -subgroup, and let $\nabla(R)$ be the FC -subalgebra of R . Then the set of algebraic elements A of $\nabla(R)$ is a locally finite algebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$ and $A/\mathfrak{J}(A)$ is commutative.*

PROOF. Since any nilpotent element of $\nabla(R)$ is central in $\nabla(R)$ by Lemma 1, one can see immediately that the set of all nilpotent elements of $\nabla(R)$ form an ideal I , and the factor algebra $\nabla(R)/I$ contains no nilpotent elements. Obviously, I is a locally finite subalgebra in $\nabla(R)$, and all idempotents of $\nabla(R)/I$ are central in $\nabla(R)/I$.

Let $x_1, x_2, \dots, x_s$ be algebraic elements of $\nabla(R)/I$. We shall prove that the subalgebra generated by $x_1, x_2, \dots, x_s$ is finite dimension.

For every x_i the subalgebra $\langle x_i \rangle_F$ of the factor algebra $\nabla(R)/I$ is a direct sum of fields

$$\langle x_i \rangle_F = F_{i1} \oplus F_{i2} \oplus \dots \oplus F_{ir_i},$$

where F_{ij} is a field and is finite dimensional over F . Choose F -basis elements u_{ijk} ($i = 1, \dots, s, j = 1, \dots, r_i, k = 1, \dots, [F_{ij} : F]$) in F_{ij} over F and denote by $w_{ijk} = 1 - e_{ij} + u_{ijk}$, where e_{ij} is the unit element of F_{ij} . Obviously, w_{ijk} is a unit in $\nabla(R)/I$. We collect in the direct summand all these units w_{ijk} for each field F_{ij} ($i = 1, \dots, s, j = 1, \dots, r_i$) and this finite subset in the group $U(\nabla(R)/I)$ is denoted by W .

Let H be the subgroup of $U(\nabla(R)/I)$ generated by W . The subgroup H of $\nabla(R)/I$ is a finitely generated FC -group, and as it is well-known, a natural number m can be assigned to H such that for any $u, v \in H$ the elements u^m, v^m are in the center $\zeta(H)$, and $(uv)^m = u^m v^m$ (see [5]).

Since H is a finitely generated group, the subgroup $S = \{v^m \mid v \in H\}$ has a finite index in H and $\{w^m \mid w \in W\}$ is a finite generated system for S . Let $t_1, t_2, \dots, t_l$ be a transversal to S in H .

Let H_F be the subalgebra of $\nabla(R)/I$ spanned by the elements of H over F . Clearly, the commutative subalgebra S_F of H_F , generated by central algebraic elements w^m ($w \in W$), is finite dimensional over F and any $u \in H_F$ can be written as

$$u = u_1 t_1 + u_2 t_2 + \dots + u_l t_l,$$

where $u_i \in S_F$. Since $t_i t_j = \alpha_{ij} t_{r(ij)}$ and $\alpha_{ij} \in S_F$, it yields that the subalgebra H_F is finite dimensional over F . Recall that

$$x_i = \sum_{j,k} \beta_{jk} w_{ijk} - \sum_{j,k} \beta_{jk} (1 - e_{ij}),$$

where $\beta_{jk} \in F$ and e_{ij} are central idempotents of $\nabla(R)/I$. The subalgebra T generated by e_{ij} ($i = 1, 2, \dots, s, j = 1, 2, \dots, r_i$) is finite dimensional over F and T is contained in the center of $\nabla(R)/I$. Therefore, x_i belongs to the sum of two subspaces H_F and T and the subalgebra of $\nabla(R)/I$ generated by H_F and T is finite dimensional over F . Since $\langle x_1, \dots, x_s \rangle_F$ is a subalgebra of $\langle H_F, T \rangle_F$, is also finite dimensional over F . We established that the set of algebraic elements of $\nabla(R)/I$ is a locally finite algebra. One can see that all the algebraic elements of $\nabla(R)$ form a locally finite algebra A (see [3], Lemma 6.4.1, p. 162). Since the radical of an algebraic algebra is a nil ideal, according to Lemma 1 we have that $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$, and $A/\mathfrak{J}(A)$ is commutative by Lemma 2. $\square$

Recall that by Neumann's Theorem [5] the set $t(\Delta U)$ of ΔU containing all elements of finite order of ΔU is a subgroup.

Theorem 2. *Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω -subgroup. Then*

1. *the elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in ΔU ;*
2. *if all elements of $\nabla(R)$ are algebraic then ΔU is nilpotent of class 2;*

3. ΔU is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2.

PROOF. It is easy to see that $\Delta U \subseteq \nabla(R)$, and any element of $t(\Delta U)$ is algebraic. According to Theorem 1 the set A of algebraic elements of $\nabla(R)$ is a subalgebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$, and $A/\mathfrak{J}(A)$ is commutative. The isomorphism

$$U(A)/(1 + \mathfrak{J}(A)) \cong U(A/\mathfrak{J}(A)),$$

implies that $(t(\Delta U)(1 + \mathfrak{J}(A)))/(1 + \mathfrak{J}(A))$ is abelian, the commutator subgroup of $t(\Delta U)$ is contained in $1 + \mathfrak{J}(A)$ and consists of unipotent elements.

By Neumann's Theorem $\Delta U/t(\Delta U)$ is abelian, therefore ΔU is a solvable group of length at most 3. $\square$

Let R be an algebra over a field F . Let m be the order of the element $g \in U(R)$ and assume that the element $1 - \alpha^m$ is a unit in F for some $\alpha \in F$. It is well-known that $g - \alpha \in U(R)$ and

$$(g - \alpha)^{-1} = (1 - \alpha^m)^{-1} \sum_{i=0}^{m-1} \alpha^{m-1-i} g^i.$$

We know that the number of solutions of the equation $x^m - 1 = 0$ in F does not exceed m . If F is an infinite field, then it follows that, there exists an infinite set of elements $\alpha \in F$ such that $g - \alpha$ is a unit. We will show that this is true for any algebraic unit.

Lemma 3. *Let $g \in U(R)$ be an algebraic element over an infinite field F . Then there are infinitely many elements α of the field F such that $g - \alpha$ is a unit.*

PROOF. Since g is an algebraic element over F , $F[g]$ is a finite dimensional subalgebra over F . Let T be the radical of $F[g]$. There exists an orthogonal system of idempotents $e_1, e_2, \dots, e_s$ such that

$$F[g] = F[g]e_1 \oplus F[g]e_2 \oplus \dots \oplus F[g]e_s,$$

and Te_i is a nilpotent ideal such that $F[g]e_i/Te_i$ is a field. It is well-known that $F[g]e_i$ is a local ring, and all elements of $F[g]e_i$, which do not belong to Te_i , are units. Moreover, if $\alpha \in F$, then

$$(2) \quad g - \alpha = (ge_1 - \alpha e_1) + (ge_2 - \alpha e_2) + \dots + (ge_s - \alpha e_s).$$

Clearly, ge_i is a unit and $ge_i \notin Te_i$ for every i . Put

$$L_i = \{ge_i - \alpha e_i \mid \alpha \in F\}.$$

Suppose that $ge_i - \beta e_i$ and $ge_i - \gamma e_i$ belong to Te_i for some $\alpha, \beta \in F$. Then

$$(ge_i - \beta e_i) - (ge_i - \gamma e_i) = (\gamma - \beta)e_i \in Te_i,$$

which is impossible for $\beta \neq \gamma$. Therefore, Te_i contains at most one element from L_i . Since $F[g]e_i$ is a local ring, all elements of the form $ge_i - \alpha e_i$ with $ge_i - \alpha e_i \notin Te_i$ are units, and there are infinitely many units of the form (2). $\square$

Lemma 4. *Let $g \in U(R)$ and $a \in R$. If $g - \alpha$, $g - \beta$ are units for some $\alpha, \beta \in F$ and $ag \neq ga$, then*

$$(g - \alpha)^{-1}a(g - \alpha) \neq (g - \beta)^{-1}a(g - \beta).$$

PROOF. Suppose that $(g - \alpha)^{-1}a(g - \alpha) = (g - \beta)^{-1}a(g - \beta)$. Then

$$(g - \alpha - (\beta - \alpha))(g - \alpha)^{-1}a = a(g - \alpha - (\beta - \alpha))(g - \alpha)^{-1}$$

$$\text{and } (1 - (\beta - \alpha)(g - \alpha)^{-1})a = a(1 - (\beta - \alpha)(g - \alpha)^{-1}).$$

Hence

$$(\beta - \alpha)(g - \alpha)^{-1}a = a(\beta - \alpha)(g - \alpha)^{-1}$$

and $(g - \alpha)^{-1}a = a(g - \alpha)^{-1}$, which provides the contradiction $ag = ga$. $\square$

Theorem 3. *Let R be an algebra over an infinite field F . Then*

1. *any algebraic unit over F belongs to the centralizer of $\nabla(R)$;*
2. *if R is generated by algebraic units over F , then $\nabla(R)$ belongs to the center of R .*

PROOF. Let $a \in \nabla(R)$, and $g \in U(R)$ be an algebraic element over F . Then by Lemma 3 there are infinitely many elements $\alpha \in F$ such that $g - \alpha$ is a unit for every α . If $[a, g] \neq 0$, then by Lemma 4 the elements of the form $(g - \alpha)^{-1}a(g - \alpha)$ are different, and a has an infinite number of conjugates, which is impossible. Therefore, g belongs to the centralizer of $\nabla(R)$.

Now, suppose that R is generated by algebraic units $\{a_j\}$ over F . Since every $w \in U(R)$ can be written as a sum of elements of the form $\alpha_i a_{i_1}^{\gamma_{i_1}} \dots a_{i_s}^{\gamma_{i_s}}$, where $\alpha_j \in F$, $\gamma_{i_j} \in \mathbb{Z}$, by the first part of this theorem w commute with elements of $\nabla(R)$. Hence $\nabla(R)$ is central in R . $\square$

Corollary. *Let R be an algebra over an infinite field F . Then*

1. *$t(\Delta U)$ is abelian and ΔU is a nilpotent group of class at most 2;*
2. *if every unit of R is an algebraic element over F , then ΔU is central in $U(R)$.*

PROOF. Clearly, all elements from $t(\Delta U)$ are algebraic and by Theorem 3 every algebraic unit belongs to the centralizer of $\nabla(R)$. Since $t(\Delta U) \subseteq \nabla(R)$, it follows that $t(\Delta U)$ is central in $\nabla(R)$. Since $\Delta U/t(\Delta U)$ is abelian, by Neumann's Theorem ΔU is a nilpotent group of class at most 2.

Let $a \in \Delta U$ and $g \in U(R)$ be an algebraic element over F . Then by Theorem 3 we get $[a, g] = 0$. Hence, if every unit of R is an algebraic element over F , then ΔU is central in $U(R)$. $\square$

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