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# On a class of modules

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**Abstract.** Let $R$ be a ring with identity and $M$ a unital right $R$-module. Let $\mathrm{Z}^*(M) = \{m \in M : mR \ll \mathrm{E}(mR)\}$. In this study we consider the property (T): For every right $R$-module $M$ with $\mathrm{Z}^*(M) = \operatorname{Rad} M$, $M$ is injective. We give a characterization of the property (T) when $R$ is a prime PI-ring. Also, over a right Noetherian ring $R$ we prove that if $R$ satisfies (T) then every right $R$-module is the direct sum of an injective module and a Max-module.

## 1. Introduction and notations

All rings have identity and all modules are unital right modules.

Let $R$ be a ring and $M$ a right $R$-module. We write $\mathrm{E}(M)$, $\operatorname{Rad} M$ and $\operatorname{Soc}(M)$ for the injective envelope, the radical and the socle of $M$, respectively. For the right annihilator of $M$ in $R$ we write $\operatorname{ann}(M)$. As usual, $\mathbb{N}$, $\mathbb{C}$ represent the sets of natural numbers and complex numbers. A submodule $N$ of $M$ is indicated by writing $N \leq M$. The notation $N \leq_e M$ is reserved for essential submodules.

Let $N$ be a submodule of $M$. $N$ is called a *small submodule* if whenever $N + L = M$ for some submodule $L$ of $M$ we have $M = L$, and in this case we write $N \ll M$. In [7] LEONARD defined a module $M$ to be *small* if it is a small submodule of some $R$-module. He showed that $M$ is small if and only if $M$ is small in its injective hull. We put

$$\mathrm{Z}^*(M) = \{m \in M : mR \text{ is small}\} \quad [5].$$

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Since $\operatorname{Rad}(M)$ is the union of all small submodules in $M$, $\mathrm{Z}^*(E) = \operatorname{Rad}(E)$ for any injective module $E$ and

$$\mathrm{Z}^*(M) = M \cap \operatorname{Rad} \mathrm{E}(M) = M \cap \operatorname{Rad} E'$$

for an injective $E' \supseteq M$.

In this note we consider the following property:

**(T)** For every right $R$-module $M$ with $\mathrm{Z}^*(M) = \operatorname{Rad}(M)$, $M$ is injective.

Clearly, semisimple rings satisfy (T). We will prove that the following are equivalent for a prime PI-ring $R$:

i) $R$ satisfies (T),

ii) For every left $R$-module with $\mathrm{Z}^*(M) = \operatorname{Rad}(M)$, $M$ is injective,

iii) $R$ is a hereditary Noetherian ring.

After that we show that over a right Noetherian ring $R$, if $R$ satisfies (T) then every right $R$-module is the direct sum of an injective module and a Max-module. Also, if $R$ is a prime right Goldie ring which is not primitive then the converse of the above result holds.

## 2. Results

We start with the following

**Lemma 1.** *For any module $M$, $\mathrm{Z}^*(M)$ is a submodule of $M$ and $\operatorname{Rad}(M) \leq \mathrm{Z}^*(M)$.*

Proof. Elementary. □

Let $R$ be a ring with identity and $M$ be a unital right $R$-module. An ideal $P$ of $R$ is called *right primitive* if there exists a simple right $R$-module $U$ such that $P$ is the annihilator of $U$ in $R$.

**Lemma 2.** *Suppose that $M = MP$ for every right primitive ideal $P$. Then $M = \operatorname{Rad} M$.*

Proof. Suppose that $M$ contains a maximal submodule $N$ and let $P = \operatorname{ann}(M/N)$. Then $M = MP \leq N$, a contradiction. □

The ring is called *right bounded* if every essential right ideal contains a two-sided ideal which is essential as a right ideal. Moreover, $R$ is *fully right bounded* if $R/P$ is a right bounded ring for every prime ideal $P$ of $R$. The abbreviation right FBN or FBN is commonly used for a right Noetherian right fully bounded or a Noetherian fully bounded ring, respectively. A ring $R$ is a *PI-ring* if $R$ satisfies a polynomial identity.

**Lemma 3.** *Suppose that $R$ is right FBN or a PI-ring. Then the ring $R/P$ is (right) Artinian for every right primitive ideal $P$ of $R$.*

Proof. (See for example [4, Proposition 8.4].) □

*Remark.* Certain group rings and certain universal enveloping algebras $R$ have the property that the ring $R/P$ is Artinian (because $R/P$ is prime, $R/P$ is right Artinian implies $R/P$ is also left Artinian) for every right primitive ideal $P$ (see [8]). Of course, simple right Noetherian rings which are not (right) Artinian, for example the Weyl algebras $\mathrm{A}_n(\mathbb{C})$ $(n \in \mathbb{N})$, do not have this property.

**Lemma 4.** *Let $R$ be a ring such that $R/P$ is an Artinian ring for every right primitive ideal $P$ of $R$. Then $M = \operatorname{Rad} M$ if and only if $M = MP$ for every right primitive ideal $P$ of $R$.*

Proof. The sufficiency follows by Lemma 2. Conversely, suppose that $M = \operatorname{Rad} M$, i.e. $M$ has no maximal submodule. Let $P$ be any right primitive ideal of $R$. Then $M/MP$ is a right module over the simple Artinian ring $R/P$ so that $M/MP$ is semisimple. Because $M$, and hence $M/MP$, does not have a maximal submodule, it follows that $M = MP$. □

Let $M$ be an injective module. Then $M = Mc$ for every regular (i.e. non-zero divisor) element $c$ in $R$. A right $R$-module $N$ is called *divisible* if $N = Nc$ for every regular $c$. Thus injective modules are divisible [9, Proposition 2.6].

**Lemma 5** [6, Proposition 3.5]**.** *Suppose that $R$ is a ring such that every divisible right $R$-module is injective. Then $R$ is right hereditary.*

*Remark.* Let $R$ be a semiprime right Goldie ring. Then any torsion free (i.e. non-singular) divisible right $R$-module is injective.

**Lemma 6.** *Let $R$ be a prime right or left Goldie ring. Let $M$ be a divisible right $R$-module. Then $M = MI$ for every non-zero ideal $I$ of $R$.*

Proof. For any non-zero ideal $I$ of $R$ there exists a regular element $c$ of $R$ such that $c \in I$. Hence $M = Mc \leq MI \leq M$, i.e. $M = MI$. □

**Lemma 7.** *Let $R$ be a prime right Noetherian ring. Then $M = MI$ for every non-zero ideal $I$ of $R$ if and only if $M = MP$ for every non-zero prime ideal $P$ of $R$.*

Proof. The necessity is clear. Conversely, suppose that $M = MP$ for every non-zero prime ideal $P$ of $R$. Let $I$ be any non-zero ideal of $R$. Then there exists a positive integer $n$ and prime ideals $P_i$ $(1 \leq \mathrm{i} \leq n)$ such that $P_1 \dots P_n \leq I \leq P_1 \cap \dots \cap P_n$. Then $M = MP_n = MP_{n-1}P_n = \dots = MP_1 \dots P_n \leq MI \leq M$, i.e. $M = MI$. □

**Lemma 8.** *Let $R$ be a left bounded left Goldie prime ring. Then the right $R$-module $M$ is divisible if and only if $M = MI$ for every non-zero ideal $I$ of $R$.*

Proof. The necessity follows by Lemma 6. Conversely, suppose that $M = MI$ for every non-zero ideal $I$ of $R$. Let $c$ be any regular element of $R$. Then there exists a non-zero ideal $J$ such that $J \leq Rc$. Now $M = MJ \leq MRc = Mc \leq M$, i.e. $M = Mc$. □

Prime PI-rings are right and left bounded and right and left Goldie, so Lemma 7 and Lemma 8 give at once:

**Corollary 9.** *Let $R$ be a prime PI-ring. Then $M$ is divisible if and only if $M = MI$ for every non-zero ideal $I$ of $R$. If, in addition, $R$ is right Noetherian, then $M$ is divisible if and only if $M = MP$ for every non-zero prime ideal $P$ of $R$.*

**Lemma 10.** *Let $R$ be a prime (right and left) FBN-ring which is not Artinian and for which every non-zero prime ideal is right primitive (maximal in this case). Then the right $R$-module $M$ satisfies $M = \operatorname{Rad} M$ if and only if $M$ is divisible.*

Proof. By Lemmas 4, 7 and 8. □

We refer to [2, Chapter 6] for the definition of *Krull Dimension*.

**Proposition 11.** *Let $R$ be a prime PI-ring of right Krull dimension 1. Then the right $R$-module $M$ satisfies $M = \operatorname{Rad} M$ if and only if $M$ is divisible.*

Proof. Suppose that $S = \operatorname{Soc} R_R \neq 0$. Then S contains a regular element $c$, because $R$ is prime right Goldie, and $R \cong cR \leq S$ gives that $R$

is right Artinian, contradicting the fact that $R$ has right Krull dimension 1. Thus $S = 0$.

Let $E$ be any essential right ideal of $R$. There exists a non-zero ideal $I$ of $R$ such that $I \leq E$. There exists a regular element $d$ such that $d \in I$. Now $R/dR$ is Artinian and hence the right $R$-module $R/I$ is Artinian (this is because the (right) Krull dimension of $R/dR$ is 0). By the Hopkins–Levitzki Theorem, the right Artinian ring $R/I$ is right Noetherian. Thus $R/E$ is a Noetherian right $R$-module. It follows that $R$ satisfies the ascending chain condition on essential right ideals and hence the ring $R/S$ is a right Noetherian ring by [2, 5.15]. Thus $R$ is a right Noetherian ring. By [8, 13.6.15 Theorem] $R$ is also left Noetherian.

It is now clear that Lemma 10 can be applied to give the result. $\square$

**Proposition 12.** *Let $R$ be a prime right Goldie ring which is not primitive. Then $Z^*(M) = M$ for every right $R$-module $M$. In addition if $R$ satisfies* (T)*, then every divisible right $R$-module is injective.*

Proof. Let $M$ be a right $R$-module, $x \in M$ and $E = \mathrm{E}(xR)$. Suppose that $E = xR + L$ for some $L \leq E$. If $x$ is not in $L$, then $E/L$ is non-zero and a cyclic module so that there exists a maximal submodule $P$ of $E$ with $L$ contained in $P$. The module $U = E/P$ is simple, and if $I$ is its annihilator in $R$ we know that $I$ is a non-zero ideal of $R$ by our hypothesis. But in this case $I$ contains a non-zero divisior by Goldie's Theorem [4, Proposition 5.9] and then $E = EI$ by [9, Proposition 2.6] so that $E = P$, a contradiction. Hence $x \in L$ and so $E = L$ and $xR$ is small. Thus $\mathrm{Z}^*(M) = M$.

Now assume that $R$ satisfies (T). Let $M$ be a divisible right $R$-module and $N$ a maximal submodule of $M$. Then $0 \neq \mathrm{ann}(M/N) \leq_e R$. There exists a non-zero regular element $d \in \mathrm{ann}(M/N)$. Now $M = Md \leq N$ and so $M = N$. Hence $\mathrm{Rad}\, M = M$. By hypothesis $M$ is injective. $\square$

*Remark.* Let $R$ be a prime PI-ring. Suppose in addition that $R$ is right hereditary. Because $R$ is right Goldie it follows that $R$ is right Noetherian [1, Corollary 8.25] and hence also left Noetherian [8, 13.6.15 Theorem]. By [8, 6.2.8 Corollary] $R$ has right Krull dimension at most 1. Note also that $R$ is left hereditary because $R$ is right and left Noetherian [1, Corollary 8.18].

**Theorem 13.** *The following are equivalent for a prime PI-ring $R$:*

(i) *For every right $R$-module $M$ with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, $M$ is injective,*

(ii) *For every left $R$-module $M$ with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, $M$ is injective,*

(iii) *$R$ is a hereditary Noetherian ring.*

Proof. (i) $\Longrightarrow$ (iii) We claim that every divisible right $R$-module is injective. Let $M$ be divisible. If $M = \mathrm{Rad}\, M$ then $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$ and hence $M$ is injective. Suppose $M \neq \mathrm{Rad}\, M$ and let $N$ be a maximal submodule of $M$. If $\mathrm{ann}(M/N) = 0$ then $R$ is primitive. By Kaplansky's Theorem, $R$ is semisimple Artinian. Hence $M$ is injective. If $\mathrm{ann}(M/N) \neq 0$, then, by Proposition 12, $M$ is injective. Thus, by Lemma 5, $R$ is right hereditary. Hence by the above remark $R$ is a hereditary Noetherian ring.

(iii) $\Longrightarrow$ (i) Let $M$ be a right $R$-module and suppose $\mathrm{Z}^*(M) = \mathrm{Rad}\, M$. By Proposition 12, $M$ has no maximal submodule. By the above remark, $R$ has (right or left) Krull dimension 1. By Proposition 11, $M$ is divisible. Hence $M$ is injective by Theorem 3.37 in [3] and Theorem 3.4 in [6].

(ii) $\Longleftrightarrow$ (iii) Symmetrical. $\square$

We call a module $M$ a *Max-module* if for every non-zero submodule $N$ of $M$, $N$ has a maximal submodule. For any module $M$ we define the radical series of $M$ to be the chain of submodules

$$M = M_0 \geq M_1 \geq \cdots \geq M_\alpha \geq M_{\alpha+1} \geq \cdots$$

where for any ordinal $\alpha \geq 0$, $\mathrm{Rad}\, M_\alpha = M_{\alpha+1}$ and $M_\alpha = \bigcap_{0 \leq \beta < \alpha} M_\beta$ if $\alpha$ is a limit ordinal. Since $M$ is a set, there exists an ordinal $\rho \geq 0$ such that $M_\rho = M_{\rho+1} = \dots$.

**Proposition 14** [10, Proposition 2.2]**.** *A module $M$ is a Max-module if and only if $M_p = 0$.*

**Theorem 15.** *Let $R$ be a right Noetherian ring. If $R$ satisfies* (T) *then every right $R$-module is the direct sum of an injective module and a Max-module.*

Proof. Let $M$ be any right $R$-module. Let $\mathcal{S}$ denote the collection of injective submodules of $M$ (note that $0 \in \mathcal{S}$). Let $\{C_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{S}$ and let $C = \bigcup C_\lambda$ ( $\lambda \in \Lambda$). Since $R$ is right Notherian, Baer's Lemma gives that $C$ is injective. Thus $C \in \mathcal{S}$. By Zorn's Lemma $\mathcal{S}$ has a maximal member $M_1$. Because $M_1$ is injective, we have $M = M_1 \oplus M_2$ for some submodule $M_2$ of $M$. Let $N$ be a non-zero submodule of $M_2$. By the choice of $M_1$, $M_1 \oplus N$, hence $N$ is not injective. Thus $\mathrm{Rad}\, N \neq \mathrm{Z}^*(N)$ by hypothesis and it follows that $N \neq \mathrm{Rad}\, N$. Therefore $N$ has a maximal submodule. Thus $M_2$ is a Max-module. $\square$

**Theorem 16.** *Let $R$ be a prime right Goldie ring which is not primitive. Assume that every right $R$-module is the direct sum of an injective module and a Max-module. Then $R$ satisfies* (T).

PROOF. Suppose that every right $R$-module is the direct sum of an injective module and a Max-module. Let $M$ be any right $R$-module such that $\mathrm{Z}^*(M) = \operatorname{Rad} M$. Then by Proposition 12, $\operatorname{Rad} M = M$. Let $M = X \oplus Y$ where $X$ is injective and $Y$ is a Max-module. Now $X \oplus Y = M = \operatorname{Rad} M = \operatorname{Rad} X \oplus \operatorname{Rad} Y$ so that $Y = \operatorname{Rad} Y$ and $Y$ does not contain a maximal submodule. This implies that $Y = 0$ and $M = X$, i.e. $M$ is injective. □

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