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A characterization of some unimodular multiplicative functions

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their sixtieth birthday

Abstract. In this paper we consider those unimodular multiplicative functions g_1, g_2 under the conditions that $g_1(an+b) - dg_2(cn)$ tends to zero in some sense. It follows from our results that the unimodular multiplicative functions g_1, g_2 satisfy the condition $g_1(an+b) - dg_2(cn) = o(1)$ if and only if there are a real number τ and unimodular multiplicative functions G_1, G_2 such that $g_1(n) = n^{i\tau}G_1(n)$, $g_2(n) = n^{i\tau}G_2(n)$ and $G_1(an+b) - d\frac{c^{i\tau}}{a^{i\tau}}G_2(cn) = 0$ hold for all positive integers n.

1. Introduction and results

An arithmetic function $g(n) \not\equiv 0$ is said to be multiplicative if $(n,m)=1$ implies that

$$g(nm) = g(n)g(m)$$

and it is completely multiplicative if this relation holds for all positive integers n and m. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the class of all complex-valued multiplicative and completely multiplicative functions, respectively. A function g is said to be unimodular if g satisfies the condition $|g(n)| = 1$ for all positive integers n. In the following we shall denote by $\mathcal{M}(1)$ and $\mathcal{M}^*(1)$ the class of all unimodular functions $g \in \mathcal{M}$ and $g \in \mathcal{M}^*$, respectively.

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The classes $\mathcal{M}(1)$ and $\mathcal{M}^*(1)$ are very important subclasses in $\mathcal{M}$ and $\mathcal{M}^*$. For each real-valued additive function f the function

$$g(n) = e^{2\pi i f(n)} \qquad (n = 1, 2, \dots)$$

belongs to $\mathcal{M}(1)$, and so results for unimodular multiplicative functions can be used to obtain information about the distribution of additive functions.

The functions of the form

$$g(n) = n^s \qquad (n = 1, 2, \dots)$$

belong to $\mathcal{M}^*$ for all fixed complex numbers s. These functions play a similar exceptional role among multiplicative functions as the functions $U \log$ among additive functions. This raises the question: Can one characterize the functions of the type $g(n) = n^s$ as multiplicative functions by imposing suitable regularity conditions on g? It turns out that this leads to problems that are much more difficult than those arising in the case of additive functions and to a large extent still unsolved.

More than 15 years ago I. KÁTAI stated as a conjecture that if $g \in \mathcal{M}(1)$ satisfies Erdős' regularity condition, i.e.

$$g(n+1) - g(n) = o(1) \quad \text{as } n \to \infty, \tag{1}$$

then

$$g(n) = n^{i\tau} \quad (n = 1, 2, \dots) \tag{2}$$

for some real number τ. This was proved by E. WIRSING in a letter to Kátai (September 3, 1984) and in a recent paper [16]. It is not hard to deduce from Wirsing's theorem that if $f, g \in \mathcal{M}(1)$, $g(n+1) - f(n) = o(1)$ as $n \to \infty$, then $f(n) = g(n) = n^{i\tau}$ $(n \in \mathbb{N})$.

It is obvious that a function of the form (2) satisfies the following weaker condition

$$\sum_{n \le x} |g(n+1) - g(n)| = o(x) \quad \text{as } x \to \infty. \tag{3}$$

It has been conjectured by I. KÁTAI [6] that the functions of the type (2) are the only unimodular multiplicative functions that satisfy (3). Although

the conjecture remains open, a few partial results are known. The problem is unsolved even in the case when g assumes only the values from ± 1. In this case (2) obviously forces $g \equiv 1$ identically, and the problem amounts to showing that

$$\limsup_{x\to\infty} \frac{1}{x} |\{n \le x : g(n+1) = -g(n)\}| > 0$$

holds, unless $g \equiv 1$. An interesting example is furnished by the Liouville function $\lambda(n)$, defined as 1 if n has an even number of prime factors, and -1 otherwise. This function belongs to $\mathcal{M}^*(1)$, but it is not known whether the last inequality holds for $g = \lambda$. The weaker result was proved by A. Hildebrand, namely he showed in [3] that if $g \in \mathcal{M}(1)$ assumes only the values ± 1, then

$$\limsup_{x\to\infty} \frac{(\log\log x)^4}{x} |\{n \le x : g(n+1) = -g(n)\}| > 0$$

holds, unless $g \equiv 1$.

It is easy to see that a function $g \in \mathcal{M}(1)$ satisfies (3) if and only if

$$\frac{1}{x} \sum_{n\le x} \overline{g(n+1)} g(n) = 1 + o(1) \quad \text{as } x \to \infty. \tag{4}$$

A. Hildebrand [5] proved that if for some real number τ

$$\operatorname{Re} \sum_p \frac{1}{p} \left(1 - \frac{g(p)}{p^{-i\tau}}\right) < \infty, \tag{5}$$

the series being taken over all primes p, then

$$\frac{1}{x} \sum_{n\le x} \overline{g(n+1)} g(n) = \prod_p F_p + o(1) \quad \text{as } x \to \infty,$$

where

$$F_p = 1 - \frac{2}{p} + 2\left(1 - \frac{1}{p}\right) \operatorname{Re} \frac{g(p)p^{-i\tau}}{p - g(p)p^{-i\tau}}.$$

Thus, the last relations with (3) jointly imply that $F_p = 1$ holds for each prime p, i.e. $g(p) = p^{i\tau}$. This shows that condition (3) along with (5) on a unimodular multiplicative function g implies (2). It would be desirable to

have a similar result for the case when (5) fails to hold, but this seems to be much more difficult. The method of Halász concerning the behaviour of the averages

$$\frac{1}{x}\sum_{n\le x} g(n)$$

for modular multiplicative function g cannot be adapted to deal with (4), since the associated Dirichlet series do not have an Euler product representation.

The question whether condition (3) alone is sufficient thus remains open. The partial results known in connection with the above mentioned Kátai' conjecture involve either stronger conditions than (3) or additional hypotheses on the function g. In a series of paper [7], [8] I. Kátai considered functions $g \in \mathcal{M}$ under the conditions that $g(n+1) - g(n)$ tends to zero in some sense. For example, it follows from Theorem 3 of I. Kátai [7] that a function $g \in \mathcal{M}(1)$ satisfying the condition

$$\sum_{n=1}^{\infty} \frac{1}{n}|g(n+1) - g(n)| < \infty$$

must be of the form (2). In a similar vein, it was shown by A. Hildebrand in [4] that if $g \in \mathcal{M}^*(1)$ satisfying (3) and in addition

$$|g(p) - 1| \le c$$

for every prime p must be of the form (2). Here c is an absolute positive constant, an admissible value of which is 10^{-3}. I. Kátai [8] proved that Hildebrand's condition for $g(p)$ can be replaced by an average condition for $g(p)$.

In 1980 J. L. Mauclaire and Leo Murata [9] proved that if $g \in \mathcal{M}(1)$ satisfies condition (3), then $g \in \mathcal{M}^*(1)$. I. Kátai [8] improved this result by showing that (3) can be replaced by a weaker condition, namely

$$\liminf_{x\to\infty} \frac{1}{x}\sum_{n\le x} |g(n+1) - g(n)| = 0.$$

In [10], improving the above result of Mauclaire and Murata, we proved that if $g \in \mathcal{M}(1)$ satisfies the condition

$$\sum_{n\le x} |g(An+B) - Dg(n)| = o(x) \quad \text{as } x \to \infty \tag{6}$$

for some positive integers A, B and a non-zero complex number D, then

$$g(p^k) = (g(p))^k \quad (k = 1, 2, \dots)$$

holds for each prime p coprime to $2AB$, furthermore in the case $(2, AB) = 1$ we also have

$$g(2^k) = \left(\frac{g(A)}{D}\right)^{k-1} g(2)^k \quad (k = 1, 2, \dots)$$

and

$$g(A)^2 = D^2.$$

In [11] we obtained similar results concerning those functions $g \in \mathcal{M}(1)$ and $g_1 \in \mathcal{M}^*(1)$, $g_2 \in \mathcal{M}(1)$ which satisfy the conditions

$$\sum_{n \le x} \frac{1}{n} |g(An + B) - Dg(n)| = o(\log x) \quad \text{as } x \to \infty \tag{7}$$

and

$$\sum_{n \le x} \frac{1}{n} |g_1(an + b) - dg_2(n)| = o(\log x) \quad \text{as } x \to \infty,$$

respectively, where A, B, a, b are fixed positive integers and D, d are non-zero complex numbers. For example, we deduced from condition (7) that there are functions $g^* \in \mathcal{M}^*$ and $G \in \mathcal{M}$ such that

$$g(n) = g^*(n)G(n), \quad G(n + B) = \frac{g(A)}{D} G(n) \quad (n = 1, 2, \dots),$$

furthermore $g^2(A) = D^2$ and in the case $2 \mid AB$ we have $g(A) = D$.

In [13] we considered the conjecture of I. Kátai concerning functions $g \in \mathcal{M}^*(1)$ satisfying (6). By using some results due to Elliott [2], we showed that if $g \in \mathcal{M}^*(1)$ satisfies (6) and there is a positive integer k such that

$$\limsup_{x \to \infty} \frac{1}{x} \Big| \sum_{n \le x} (g(n))^k \Big| > 0,$$

then there are a real constant τ and a function $G \in \mathcal{M}^*(1)$ for which

$$g(n) = n^{i\tau} G(n) \quad \text{and} \quad [G(n)]^k = 1$$

hold for all positive integers n, moreover

$$\sum_{n\le x} |G(n+1) - G(n)| = o(x) \quad \text{as } x \to \infty.$$

In this paper we shall improve these results by proving the following theorems.

Theorem 1. *Let a, b, c be positive integers and let d be a non-zero complex number. Then functions $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy the condition*

$$\sum_{n\le x} \frac{1}{n} |g_1(an+b) - dg_2(cn)| = o(\log x) \quad \text{as } x \to \infty \tag{8}$$

if and only if there are functions $g^* \in \mathcal{M}^*(1)$ and $G_1, G_2 \in \mathcal{M}(1)$ such that

$$g_1(n) = g^*(n)G_1(n), \quad g_2(n) = g^*(n)G_2(n) \quad (n = 1, 2, \dots), \tag{9}$$

$$G_1(an+b) - d\frac{g^*(c)}{g^*(a)} G_2(cn) = 0 \qquad (n = 1, 2, \dots) \tag{10}$$

and

$$\sum_{n\le x} \frac{1}{n} |g^*(n+1) - g^*(n)| = o(\log x) \quad \text{as } x \to \infty. \tag{11}$$

Remark. It follows from our proof that

$$G_1(n) = \chi_{2abc}(n) \quad \text{if } (n, 2abc) = 1$$

and

$$G_2(n) = G_2\left[(n, b^2c^2N_2)\right] \quad (n = 1, 2, \dots),$$

where χ_{2abc} denotes a Dirichlet character (mod $2abc$) and $N_2 \in \{1, 2\}$ satisfying $(2, abN_2 + 1) = 1$.

Theorem 2. *Let a, b, c be positive integers and let d be a non-zero complex number. Then functions $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy the condition*

$$\sum_{n\le x} |g_1(an+b) - dg_2(cn)| = o(x) \quad \text{as } x \to \infty \tag{12}$$

if and only if there are functions $g^* \in \mathcal{M}^*(1)$, $G_1 \in \mathcal{M}(1)$ and $G_2 \in \mathcal{M}(1)$ such that (9), (10) *and the relation*

$$\sum_{n\le x} |g^*(n+1) - g^*(n)| = o(x) \quad \text{as } x \to \infty \tag{13}$$

hold.

Applying the above results we can extend the result of E. Wirsing [16] as follows:

Theorem 3. *Let a, b, c be positive integers and let d be a non-zero complex number. Then functions $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy the condition*

$$g_1(an+b) - dg_2(cn) = o(1) \quad \text{as } n \to \infty \tag{14}$$

if and only if there are a real number τ and functions $G_1 \in \mathcal{M}(1)$, $G_2 \in \mathcal{M}(1)$ such that

$$g_1(n) = n^{i\tau} G_1(n), \quad g_2(n) = n^{i\tau} G_2(n)$$

and

$$G_1(an+b) - d\frac{c^{i\tau}}{a^{i\tau}} G_2(cn) = 0$$

hold for all positive integers n.

We note that by writing a multiplicative function $g \in \mathcal{M}(1)$ in the form $g = e^{2\pi i f}$, where f is a real-valued additive function, one can reformulate results involving unimodular multiplicative functions in term of real-valued additive functions reduced modulo 1. For example, by using the fact

$$\|u\| \ll |e^{2\pi i u} - 1| \ll \|u\|,$$

where $\|u\|$ denotes the distance of a real number u to the nearest integer, the following corollaries follow directly from Theorem 1 and Theorem 3.

Corollary 1. *Let a, b, c be positive integers and let d be a real number. Then the real-valued additive functions f_1 and f_2 satisfy the condition*

$$\sum_{n\le x}\frac{1}{n}\|f_1(an+b)-f_2(cn)-d\| = o(\log x) \quad \text{as } x\to\infty$$

if and only if there are real-valued functions $f^*\in\mathcal{A}^*$ and $F_1, F_2\in\mathcal{A}$ such that

$$\|f_1(n)-f^*(n)-F_1(n)\|=0, \quad \|f_2(n)-f^*(n)-F_2(n)\|=0,$$
$$\|F_1(an+b)-F_2(cn)-d-f^*(c)+f^*(a)\|=0$$

hold for all positive integers n and

$$\sum_{n\le x}\frac{1}{n}\|f^*(n+1)-f^*(n)\| = o(\log x) \quad \text{as } x\to\infty.$$

Corollary 2. *Let a, b, c be positive integers and let d be a real number. Then real-valued additive functions f_1 and f_2 satisfy the condition*

$$\|f_1(an+b)-f_2(cn)-d\| = o(1) \quad \text{as } n\to\infty$$

if and only if there are a real number τ and real-valued additive functions F_1, F_2 such that

$$\|f_1(n)-\tau\log n-F_1(n)\|=0, \quad \|f_2(n)-\tau\log n-F_2(n)\|=0$$

and

$$\left\|F_1(an+b)-F_2(cn)-d+\tau\log\left(\frac{a}{c}\right)\right\|=0$$

hold for all positive integers n.

Theorem 4. *Let a, b, c be positive integers and let d be a non-zero complex number. Assume that $g_1\in\mathcal{M}^*(1)$ and $g_2\in\mathcal{M}(1)$ satisfy (12), i.e.*

$$\sum_{n\le x}|g_1(an+b)-dg_2(cn)| = o(x) \quad \text{as} \quad x\to\infty.$$

If there is a positive integer k such that

$$\limsup_{x\to\infty}\frac{1}{x}\Big|\sum_{n\le x}(g_1(n))^k\Big|>0, \tag{15}$$

then there are a real constant τ *and functions* $G, G_2 \in \mathcal{M}^*(1)$ *such that*

$$g_1(n) = n^{i\tau}G(n), \quad g_2(n) = n^{i\tau}G(n)G_2(n),$$

and

$$[G(n)]^k = 1, \quad G_2(cn) = G_2(c)$$

hold for all positive integers n*, moreover*

$$\sum_{n\le x} |G(n+1) - G(n)| = o(x) \quad \text{as } x \to \infty.$$

Remark. We think that (6) with a function $g \in \mathcal{M}^*(1)$ implies (15), but we are unable to prove it presently. By writing a multiplicative function $g \in \mathcal{M}^*(1)$ in the form $g = e^{2i\pi f}$, where f is a real-valued additive function, then it is known from Chapter 8 of [1] that there are two possibilities: Either (15) holds for some positive integer k or $f(n)$ is uniformly distributed (mod 1).

2. Auxiliary results

For the proof of our theorems we need some lemmas. Let C_A denote the product of all distinct prime divisors of C which are prime to A and $E(n) = E_B(n)$ denotes the product of all prime power factors of B composed from the prime divisors of n.

In this section we consider those functions $g \in \mathcal{M}(1)$ for which the relation

$$\sum_{n\le x} \frac{1}{n} |g(An+B) - Dg(Cn)| = o(\log x) \quad \text{as } x \to \infty \tag{16}$$

holds for some positive integers A, B, C and for a non-zero complex number D.

Lemma 1. *Assume that* $g \in \mathcal{M}(1)$ *satisfies* (16) *for some positive integers* A, B, C *and for a non-zero complex constant* D*. Then for positive integers* k *and* Q *satisfying*

$$(E(CQ) + B, R_k(AC_AQ), 2) = 1, \tag{17}$$

we have

$$g(BCC_AQ^k) = W^{k-1}\frac{g(BCC_AQ)^k}{g(BCC_A)^{k-1}}, \tag{18}$$

where

$$W = W(A,B,C,D,g) := D\frac{g(CC_AE(C))g(BCC_A)}{g(ACC_A^2E(C))g(B)}. \tag{19}$$

Furthermore

$$W^2 = 1 \tag{20}$$

and

$$W = 1 \quad \textit{if } 2 \mid ABC. \tag{21}$$

Remark. It is obvious that for each fixed real number τ the function

$$g(n) = (-1)^{n-1}n^{i\tau} \quad (n = 1, 2, \dots)$$

belongs to $\mathcal{M}(1)$ and it satisfies (16) with $A = B = C = -D = 1$. In this case, by (19) we have

$$W(A,B,C,D,g) = -1.$$

In other words, from (16) not always follows that $W(A,B,C,D,g) = 1$.

Proof. We shall use an argument similar to that of the proof of Lemma 1 of [14]. For each positive integer Q we define the sequence

$$R = R(AC_AQ) = \{R_k(AC_AQ)\}_{k=1}^{\infty}$$

by the initial term $R_1(AC_AQ) = 1$ and by the formula

$$R_k(AC_AQ) = 1 + AC_AQ + \dots + (AC_AQ)^{k-1}$$

for all integers $k \geq 2$. Let

$$T_k(n,Q) = (AC_AQ)^kE(CQ)n + BR_k(AC_AQ).$$

Since

$$T_{k+1}(n,Q) = AC_AQT_k(n,Q) + B$$

and

$$(CC_AQE(CQ), T_k(n,Q)/E(CQ)) = 1$$

(see (8) and (9) in [14]), using the fact $|g| \equiv 1$, one can deduce from (16) that

$$\sum_{n\le x} \frac{1}{n} \left| g(T_k(n,Q)) - D\Delta(Q)^{k-1} g(CC_AQE(CQ)n) \right| = o(\log x) \tag{22}$$

holds for all positive integers k and Q, where

$$\Delta(Q) := D\frac{g(CC_AQE(CQ))}{g(E(CQ))} = D\frac{g(BCC_AQ)}{g(B)}.$$

Let k, Q and P be positive integers. By considering

$$n := PR_k(AC_AQ)\{APCQR_k(AC_AQ)m + 1\}$$

and taking into account (22), it follows that if positive integers k, P and Q satisfy the conditions

$$(P, R_k(AC_AQ), 2B) = (PE(CQ) + B, R_k(AC_AQ), 2) = 1, \tag{23}$$

then

$$g\left(A^{k-1}CC_A^kQ^kPE(CQ)\right) = \Delta(Q)^{k-1} g(CC_AQPE(CQ)). \tag{24}$$

We shall prove that

$$g\left(ACC_A^2PE(C)\right) = \frac{g(ACC_A^2E(C))}{g(CC_AE(C))} g(CC_APE(C)) \tag{25}$$

holds for all positive integers P.

We first consider the case when $2 \mid ABC$. In this case (23) holds if and only if $(P, R_k(AC_AQ), B) = 1$. Thus, by applying (24) with $k = 2$ and $Q = 1$, we have

$$g(ACC_A^2PE(C)) = \Delta(1)g(CC_APE(C)) \quad \text{if } (P, 1 + AC_A, B) = 1, \tag{26}$$

and so

$$g(ACC_A^2E(C)) = \Delta(1)g(CC_AE(C)). \tag{27}$$

Thus, by using (26) and the multiplicativity of g, it is obvious that

$$g\left(ACC_A^2PE(C)\right) = \Delta(1)g(CC_APE(C)) \quad (P = 1, 2, \dots).$$

From (27) we have

$$\Delta(1) = \frac{g(ACC_A^2E(C))}{g(CC_AE(C))},$$

which proves (25) in case $2 \mid ABC$.

Suppose now that $(2, ABC) = 1$. Since (23) holds for every even positive integer Q with

$$(P, R_k(AC_AQ), B) = 1,$$

by applying (24) with $k = Q = 2$ and using $E(2C) = E(C)$, we have

$$g\left(ACC_A^2 2^2 PE(C)\right) = \Delta(2)g(CC_A 2PE(C)) \quad \text{if } (P, 1 + 2AC_A, B) = 1.$$

This with the multiplicativity of g implies that

$$g\left(ACC_A^2 2^2 PE(C)\right) = \Delta(2)g(CC_A 2PE(C)) \quad (P = 1, 2, \dots),$$

consequently

$$g\left(ACC_A^2PE(C)\right) = \Delta(1)\frac{(g(2))^2}{g(2^2)}g(CC_APE(C))$$

holds for every odd positive integer P. This with the multiplicativity of g and the fact $(2, ABC) = 1$ shows that

$$g\left(ACC_A^2PE(C)\right) = \Delta(1)\frac{(g(2))^2}{g(2^2)}g(CC_APE(C)) \quad (P = 1, 2, \dots). \tag{28}$$

Applying (28) with $P = 1$, we have

$$\Delta(1)\frac{(g(2))^2}{g(2^2)} = \frac{g(ACC_A^2E(C))}{g(CC_AE(C))}.$$

This with (28) completes the proof of (25) in the case $(2, ABC) = 1$. Thus, we have proved (25).

We shall apply (23), (24) and (25) to get (18). Indeed, it follows from (23) and (24) that if positive integers k, Q satisfy (17), then

$$g\left(A^{k-1}CC_A^kQ^kE(CQ)\right) = \Delta(Q)^{k-1}g(CC_AQE(CQ)) = D^{k-1}\frac{g(BCC_AQ)^{k-1}}{g(B)^{k-1}}g(CC_AQE(CQ)).$$

Thus, if (17) holds, then from (19), (25) and (29) we have

$$g(BCC_AQ^k) = \left(D\frac{g(CC_AE(C))g(BCC_A)}{g(ACC_A^2E(C))g(B)}\right)^{k-1}\frac{g(BCC_AQ))^k}{g(BCC_A)^{k-1}} = W^{k-1}\frac{g(BCC_AQ))^k}{g(BCC_A)^{k-1}},$$

which proves (18).

In order to see (20) and (21), we shall apply (18). One can check that condition (17) is satisfied in the following cases: (i) $Q = 1$ and $k = 3$; (ii) $Q = 1$ and $k = 2$ if $2 \mid ABC$. Thus, from (18) we get $W^2 = 1$, furthermore $W = 1$ if $2 \mid ABC$, which prove (20) and (21).

The proof of Lemma 1 is finished. □

Lemma 2. *Assume that $g \in \mathcal{M}^*(1)$ satisfies (16) for some positive integers A, B, C and for a non-zero complex constant D. Then*

$$g(A) = Dg(C) \tag{30}$$

and

$$\sum_{n\le x}\frac{1}{n}|g(n+1) - g(n)| = o(\log x) \quad \text{as } x \to \infty.$$

Proof. First we note that if $g \in \mathcal{M}^*(1)$ satisfies (16), then as we have seen in the proof of Lemma 1, (18) holds for all positive integers k and Q, i.e.

$$W(A, B, C, D, g) = 1.$$

This with (19) shows that $g(A) = Dg(C)$, as asserted in Lemma 2. Thus, (30) holds. Here we prove (30) directly.

Since $g \in \mathcal{M}^*(1)$ and

$$(A^2Nm + B)(AN + 1) = AN[A(AN + 1)m + B] + B,$$

we have

$$\begin{aligned}&\{g(A^2Nm+B)-Dg(ACNm)\}g(AN+1)\\&\quad=\{g(AN[A(AN+1)m+B]+B)-Dg(CN)g[A(AN+1)m+B]\}\\&\qquad+Dg(CN)\{g[A(AN+1)m+B]-Dg(C)g(AN+1)g(m)\}\\&\qquad+Dg(CN)g(AN+1)\{Dg(C)-g(A)\}g(m).\end{aligned}$$

This, by using (16) and the fact $|g|\equiv 1$, implies that

$$|Dg(C)-g(A)|\sum_{m\le x}\frac{1}{m}=o(\log x),$$

which with a known asymptotic relation

$$\sum_{m\le x}\frac{1}{m}=\log x+\text{constant}+O\left(\frac{1}{x}\right)$$

proves (30).

By using (30), from (16) we can assume that $g\in\mathcal{M}^*(1)$ satisfies

$$\sum_{n\le x}\frac{1}{n}|g(An+B)-g(An)|=o(\log x)\quad\text{as } x\to\infty. \tag{31}$$

We denote by J_g the set of those pairs (Q,R) of positive integers for which

$$\sum_{n\le x}\frac{1}{n}|g(Qn+R)-g(Qn)|=o(\log x)\quad\text{as } x\to\infty.$$

We first prove that

$$(Q,R)\in J_g\quad\text{if } 0<R<Q. \tag{32}$$

By using the same method that was applied in the proof of the first part of Lemma 2 in [14], from (31) and using $|g|\equiv 1$, it follows that the following assertions hold:

(a) $(Q,1)\in J_g$ if $(q,1)\in J_g$ and $Q\ge q$

(b) $(Q,R)\in J_g$ if $(q,1)\in J_g$, $q\ge 2$ and $0<R<Q/(q-1)$

(c) $(h,1)\in J_g$ if $(h+1,1)\in J_g$ and $h\ge 2$.

From (31), we have $(A, B) \in J_g$ and so $(A, 1) \in J_g$. If $A = 1$, then the assertion of Lemma 2 holds. If $A \geq 2$, then by using (c) one can deduce that $(2, 1) \in J_g$, and so by applying (b) with $q = 2$, it follows that $(Q, R) \in J_g$ for all integers $0 < R < Q$. This completes the proof of (32).

Let $Q \geq 2$ be a fixed positive integer. For each integer $\gamma \geq 0$ let

$$\mathcal{B}_\gamma := \{n \in \mathbb{N} : Q^\gamma \parallel (n+1)\}$$

and

$$S_\gamma(x) := \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_\gamma}} \frac{1}{n} |g(n+1) - g(n)|.$$

By using (31), one can get from (32) that

$$S_0(x) := \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_0}} \frac{1}{n} |g(n+1) - g(n)| = o(1) \quad \text{as } x \to \infty. \tag{33}$$

Thus, it follows from (32) and (33) that

$$\begin{aligned} S_\gamma(x) &:= \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_\gamma}} \frac{1}{n} |g(n+1) - g(n)| \\ &= \frac{1}{\log x} \sum_{\substack{m+1 \leq (x+1)/Q^\gamma \\ m \in \mathcal{B}_0}} \frac{1}{m} |g(Q^\gamma) g(m+1) - g(Q^\gamma m + Q^\gamma - 1)| \\ &\leq \frac{1}{\log x} \sum_{\substack{m+1 \leq (x+1)/Q^\gamma \\ m \in \mathcal{B}_0}} \frac{1}{m} |g(Q^\gamma)[g(m+1) - g(m)]| \\ &\quad + \frac{1}{\log x} \sum_{\substack{m+1 \leq (x+1)/Q^\gamma \\ m \in \mathcal{B}_0}} \frac{1}{m} |g(Q^\gamma m + Q^\gamma - 1) - g(Q^\gamma m)|, \end{aligned}$$

and so

$$S_\gamma(x) = o(1) \quad \text{as } x \to \infty. \tag{34}$$

Relations (33) and (34) together with $|g| \equiv 1$ imply that for each positive integer M, we have

$$\frac{1}{\log x}\sum_{n\le x}\frac{1}{n}|g(n+1)-g(n)|$$
$$\le \sum_{0\le j\le M-1} S_j(x) + \frac{1}{\log x}\sum_{\substack{n\le x\\ Q^M|(n+1)}}\frac{2}{n} \le o(M) + \frac{2}{Q^M},$$

and so

$$\limsup_{x\to\infty}\frac{1}{\log x}\sum_{n\le x}\frac{1}{n}|g(n+1)-g(n)| \ll Q^{-M}.$$

This with $M\to\infty$ shows that

$$\frac{1}{\log x}\sum_{n\le x}\frac{1}{n}|g(n+1)-g(n)| = o(1) \quad \text{as } x\to\infty,$$

as asserted in Lemma 2. The proof of Lemma 2 is finished. □

Lemma 3. *Let A, B, C be positive integers and let D be a non-zero complex number. If $g\in\mathcal{M}^*(1)$ satisfies the condition*

$$\sum_{n\le x}|g(An+B)-Dg(Cn)| = o(x) \quad \text{as } x\to\infty,$$

then we have

$$\sum_{n\le x}|g(n+1)-g(n)| = o(x) \quad \text{as } x\to\infty.$$

Proof. We use an argument similar to that of the proof of Lemma 2 and in this way we infer by the assumption of Lemma 3 that

$$\sum_{n\le x}|g(Qn+R)-g(Qn)| = o(x) \quad \text{as } x\to\infty$$

holds for all positive integers Q, R satisfying $0 < R < Q$. Let

$$\mathcal{B}_\gamma := \{n \in \mathbb{N} : Q^\gamma \parallel (n+1)\}$$

and

$$T_\gamma(x) := \frac{1}{x} \sum_{\substack{n \le x \\ n \in \mathcal{B}_\gamma}} |g(n+1) - g(n)|.$$

One can deduce as in the proof of Lemma 2 that for each integer $\gamma \ge 0$ we have

$$T_\gamma(x) = o(1),$$

consequently

$$\frac{1}{x} \sum_{n \le x} |g(n+1) - g(n)| = o(1) \quad \text{as } x \to \infty.$$

So, the proof of Lemma 3 is finished. □

3. Proof of Theorem 1

We suppose that functions $g^* \in \mathcal{M}^*(1)$, $G_1 \in \mathcal{M}(1)$ and $G_2 \in \mathcal{M}(1)$ satisfy (10) and (11). Then it is easy to see that the functions

$$g_1 := g^* G_1, \qquad g_2 = g^* G_2$$

are unimodular multiplicative and the condition

$$g_1(an+b) - dg_2(cn) = G_1(an+b)\{g^*(an+b) - g^*(an)\}$$

holds for all positive integers n. This with (11) shows that (8) holds. Thus, we have proved the sufficiency part of Theorem 1.

In the following we shall prove the necessity part of the theorem. Assume that $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy (8) for some positive integers a, b, c and a non-zero complex constant d, i.e.

$$\sum_{n \le x} \frac{1}{n} |g_1(an+b) - dg_2(cn)| = o(\log x) \quad \text{as } x \to \infty. \tag{35}$$

It is obvious that for each positive integer N

$$(abN+1, a(abN+1)n+b) = 1$$

and

$$(abN+1)(a(abN+1)n+b) = a[(abN+1)^2 n + b^2 N] + b$$

hold for every positive integer n. Thus, by using the multiplicativity of g_2, we get

$$\begin{aligned}&\{g_2[(abN+1)^2 cn + b^2 cN] - g_1(abN+1)g_2[(abN+1)cn]\}d\\ &\quad = -\{g_1[(abN+1)(a(abN+1)n+b)] - dg_2[(abN+1)^2 cn + b^2 cN]\}\\ &\qquad + \{g_1[a(abN+1)n+b] - dg_2[(abN+1)cn]\}g_1(abN+1),\end{aligned}$$

which, using (35) and the facts $|g_1| \equiv 1$, $d \neq 0$, implies that

$$\sum_{n\le x} \frac{1}{n}\Big|g_2[(abN+1)^2 cn + b^2 cN] - g_1(abN+1)g_2[(abN+1)cn]\Big| = o(\log x). \tag{36}$$

Applying Lemma 1 with

$$A = (abN+1)^2 c, \quad B = b^2 cN \quad \text{and} \quad C = (abN+1)c,$$

and using the fact $C_A = 1$, it follows from (36) that if positive integers k, Q and N satisfy

$$(E((abN+1)cQ) + b^2 cN, R_k((abN+1)^2 cQ), 2) = 1, \tag{37}$$

then

$$g_2\left[b^2c^2(abN+1)NQ^k\right] = W(N)^{k-1}\frac{g_2(b^2c^2(abN+1)NQ)^k}{g_2(b^2c^2(abN+1)N)^{k-1}}, \tag{38}$$

where $W(N) = W(a,b,c,d,N,g_2)$ is defined in (19) and in our case $W(N)$ satisfies the conditions

$$W(N)^2 = 1 \tag{39}$$

and

$$W(N) = 1 \quad \text{if } 2 \mid (abN+1)bcN. \tag{40}$$

For each positive integer m, let N_m be the smallest positive integer for which $(abN_m + 1, m) = 1$. It is obvious that $N_p \in \{1, 2\}$ for all primes p.

We note that if $Q = 2m$, then (37) holds for all positive integers k, m and N. Thus, by applying (38) with $Q = 2$,$N = N_p$ and $Q = 2p$, $N = N_p$, we have

$$g_2\left[b^2c^2(abN_p+1)N_p2^k\right] = W(N_p)^{k-1}\frac{g_2(b^2c^2(abN_p+1)N_p2)^k}{g_2(b^2c^2(abN_p+1)N_p)^{k-1}}$$

and

$$g_2\left[b^2c^2(abN_p+1)N_p2^kp^k\right] = W(N_p)^{k-1}\frac{g_2(b^2c^2(abN_p+1)N_p2p)^k}{g_2(b^2c^2(abN_p+1)N_p)^{k-1}}.$$

Since $(abN_p + 1, p) = 1$, these relations imply that if $p \neq 2$ is a prime number, then

$$g_2\left(b^2c^2N_pp^k\right) = \frac{g_2(b^2c^2N_pp)^k}{g_2(b^2c^2N_p)^{k-1}}. \tag{41}$$

On the other hand, by applying (38) with $Q = 2$ and $N = N_2$, we have

$$g_2\left(b^2c^2(abN_2+1)N_22^k\right) = W(N_2)^{k-1}\frac{g_2(b^2c^2(abN_2+1)N_22)^k}{g_2(b^2c^2(abN_2+1)N_2)^{k-1}}. \tag{42}$$

Let $W = W(N_2)$. Now we define the completely multiplicative function g^* for each prime p as follows:

$$g^*(p) = \begin{cases} \dfrac{g_2(b^2c^2N_pp)}{g_2(b^2c^2N_p)} & \text{if } (p,2) = 1 \\ W\dfrac{g_2(2b^2c^2(abN_2+1)N_2)}{g_2(b^2c^2(abN_2+1)N_2)} & \text{if } p = 2. \end{cases} \tag{43}$$

Let

$$g_2(n) := g^*(n)G_2(n) \quad (n = 1, 2, \dots). \tag{44}$$

Then, it follows from (41)–(44) that

$$G_2\left(b^2c^2N_pp^k\right) = G_2\left(b^2c^2N_p\right) \quad (p \neq 2)$$

and

$$G_2\left(b^2c^2(abN_2+1)N_2 2^k\right) = WG_2\left(b^2c^2(abN_2+1)N_2\right)$$

hold for all positive integers k. Since $(N_2N_p, p) = 1$ for odd primes p and $(abN_2+1, 2) = 1$, the last relations with (39) and (40) imply

$$G_2(n) = W^{n-1}G_2\left[(n, b^2c^2N_2)\right] \quad (n = 1, 2, \dots). \tag{45}$$

We shall prove

$$g_1(abcN_2M+1) = W^M g^*(abcN_2M+1) \tag{46}$$

holds for all positive integers M and g^* satisfies (11).

Let M be a positive integer. By using (39) and (40), it follows from (44) and (45) that

$$\begin{aligned} &g_2\left[(abcN_2M+1)^2bcN_2m + b^2c^2N_2M\right] \\ &\quad = W^{m+M-1}g^*(bcN_2)g^*\left[(abcN_2M+1)^2m + bcM\right]G_2[bcN_2(m, bc)] \end{aligned}$$

and

$$\begin{aligned} &g_2[(abcN_2M+1)bcN_2m] \\ &\quad = W^{m-1}g^*\left[bcN_2(abcN_2M+1)\right]G_2[bcN_2(m, bc)]g^*(m). \end{aligned}$$

Applying (36) with $n = bN_2m$ and $N = cN_2M$ and using the above relations, we have

$$\sum_{n\le x}\frac{1}{n}\Big|g^*[(abcN_2M+1)^2m + bcN] - \\ -W^M g_1(abcN_2M+1)g^*(abcN_2M+1)g^*(m)\Big| = o(\log x),$$

which, using Lemma 2, proves (11) and (46).

Let

$$g_1(n) = g^*(n)G_1(n) \qquad (n = 1, 2, \dots). \tag{47}$$

We shall prove that the functions $g^*(n)$, $G_1(n)$ and $G_2(n)$ satisfy (10) and

$$|g^*(n)| = |G_1(n)| = |G_2(n)| = 1 \quad (n = 1, 2, \dots). \tag{48}$$

First we prove (48). Since $g_1, g_2 \in \mathcal{M}(1)$, $|W| = 1$ and $g_i = g^* G_i$ $(i = 1, 2)$, we have

$$|g^*(n)| \,.\, |G_1(n)| = |g^*(n)| \,.\, |G_2(n)| = 1 \quad (n = 1, 2, \dots). \tag{49}$$

Assume that there is a positive integer N such that

$$|g^*(N)| \neq 1.$$

By using (45), it follows that there are positive constants M_1 and M_2 such that

$$M_1 < |G_2(n)| < M_2 \quad (n = 1, 2, \dots).$$

Thus, we get from (49) that for all positive integers k

$$\frac{1}{M_2} < |g^*(N^k)| = |g^*(N)|^k = \frac{1}{|G_2(N^k)|} < \frac{1}{M_1},$$

which is impossible in the case $|g^*(N)| \neq 1$. This implies that $|g^*| \equiv |G_2| \equiv 1$. Thus, (49) is proved.

It remains to prove (10). Since

$$g_1(an+b) - dg_2(cn) = g^*(an+b)G_1(an+b) - dg^*(cn)G_2(cn)$$
$$= G_1(an+b)\{g^*(an+b) - g^*(an)\} + g^*(an)\Big\{G_1(an+b) - d\frac{g^*(c)}{g^*(a)}G_2(cn)\Big\}$$

and

$$\sum_{n \le x} \frac{1}{n} |g^*(n+1) - g^*(n)| = o(\log x),$$

as we have proved above, one can deduce from (35) and (49) that

$$\sum_{n \le x} \frac{1}{n} \Big| G_1(an+b) - d\frac{g^*(c)}{g^*(a)} G_2(cn) \Big| = o(\log x). \tag{50}$$

It is easy to check from (39), (40), (45), (46) and (47) that $G_1(2abcN_2m + 1) = 1$ and

$$G_2\left[(aN+b)^2 bc^2 N_2 m + Nc\right] = G_2(cN)$$

hold for all positive integers m. Let N be a positive integer. Then it follows from the above relations that

$$\begin{aligned} &G_1(aN+b) - d\frac{g^*(c)}{g^*(a)}G_2(cN) \\ &\quad = G_1(aN+b)G_1(2abcN_2m+1) - d\frac{g^*(c)}{g^*(a)}G_2(cN) \\ &\quad = G_1[a(2(aN+b)bcN_2m+N)+b] \\ &\qquad - d\frac{g^*(c)}{g^*(a)}G_2[2(aN+b)bc^2N_2m+cN]. \end{aligned}$$

Applying (50) with $n = 2(aN+b)bcN_2m + N$, we infer from the above relation that

$$\Big|G_1(aN+b) - d\frac{g^*(c)}{g^*(a)}G_2(cN)\Big| \sum_{m\le x}\frac{1}{m} = o(\log x),$$

which implies

$$G_1(aN+b) - d\frac{g^*(c)}{g^*(a)}G_2(cN) = 0.$$

Since N is an arbitrary positive integer, the last relation shows that (10) holds. Thus, we have proved the necessity part of Theorem 1, and so the proof of Theorem 1 is finished.

4. Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2. Assume that functions $g^* \in \mathcal{M}^*(1)$ and $G_1, G_2 \in \mathcal{M}(1)$ satisfy (10) and (13). Then the function $g_1 = g^*G_1$ and $g_2 = g^*G_2$ are unimodular multiplicative functions. One can show that (12) is satisfied.

Conversely, assume that $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy (12). Then condition (8) of Theorem 1 also holds. Thus, by using Theorem 1, there are functions $g^* \in \mathcal{M}^*(1)$ and $G_1, G_2 \in \mathcal{M}(1)$ such that $g_1 = g^*G_1$, $g_2 = g^*G_2$ and (10) holds, consequently

$$g_1(an+b) - dg_2(cn) = G_1(an+b)\{g^*(an+b) - g^*(an)\}.$$

This with Lemma 3 proves (13). So, the necessity part of Theorem 2 is proved. This completes the proof of Theorem 2. □

PROOF of Theorem 3. The sufficiency part of the theorem is obvious.

Assume that functions $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy (14). First we note, by using Theorem 1 and Lemma 2, that there are functions $g^* \in \mathcal{M}^*(1)$ and $G_1, G_2 \in \mathcal{M}(1)$ such that

$$g_1 = g^* G_1, \qquad g_2 = g^* G_2 \tag{51}$$

and

$$G_1(an+b) - d\frac{g^*(c)}{g^*(a)} G_2(cn) = 0 \quad (n = 1, 2, \dots). \tag{52}$$

We shall prove that there is a real number τ such that

$$g^*(n) = n^{i\tau} \qquad (n = 1, 2, \dots).$$

First we note by using (14), (51) and (52) that

$$g_1(an+b) - dg_2(cn) = G_1(an+b)\{g^*(an+b) - dg^*(cn)\} = o(1) \quad \text{as } n \to \infty,$$

and so

$$g^*(an+b) - dg^*(cn) = o(1) \quad \text{as } n \to \infty.$$

This, using Lemma 2 and the complete multiplicativity of g^*, shows that

$$g^*(an+1) - g^*(an) = o(1) \quad \text{as } n \to \infty. \tag{53}$$

We shall deduce from (53) that for each positive K we have

$$g^*(an+K) - g^*(an) = o(1) \quad \text{as } n \to \infty \tag{54}$$

It is obvious from (53) that (54) is valid for $K = 1$. Assume that (54) holds for K. We shall prove that (54) is satisfied for $K+1$.

Since

$$(an+K)(an+1) = a(an+K+1)n + K$$

and $g^* \in \mathcal{M}^*$, we have

$$\begin{aligned} g^*(an)\{g^*(an+K+1) - g^*(an)\} &= g^*(an+1)\{g^*(an+K) - g^*(an)\} \\ &\quad - \{g^*[a(an+K+1)n + K] - g^*[a(an+K+1)n]\} \\ &\quad + g^*(an)\{g^*(an+K) - g^*(an)\}. \end{aligned}$$

This with our assumption implies that (54) holds for $K+1$. Thus, we have proved (54).

Finally, by applying (54) with $K = a$ and using the complete multiplicativity of g^*, it follows that

$$g^*(a)\{g^*(n+1) - g^*(n)\} = o(1) \quad \text{as } n \to \infty.$$

From this and using the result of E. Wirsing, it follows that there is a real number τ such that $g^*(n) = n^{i\tau}$. This with (51) and (52) completes the proof of the necessity part of Theorem 3. The proof of Theorem 3 is finished. □

5. Proof of Theorem 4

Let a, b, d be positive integers and let d be a non-zero complex number. Assume that $g_1 \in \mathcal{M}^*(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy (12), i.e.

$$\sum_{n \le x} |g_1(an+b) - dg_2(cn)| = o(x) \quad \text{as } x \to \infty.$$

First we note, by using the fact $g_1 \in \mathcal{M}^*(1)$ and Theorem 2, that there is a function $G_2 \in \mathcal{M}(1)$ such that $g_2 = g_1 G_2$,

$$G_2(cn) = \frac{g_1(a)}{dg_1(c)} \quad (n = 1, 2, \dots)$$

and

$$\sum_{n \le x} |g_1(n+1) - g_1(n)| = o(x) \quad \text{as } x \to \infty.$$

These imply that

$$G_2(cn) = G_2(c) = \frac{g_1(a)}{dg_1(c)} \quad (n = 1, 2, \dots)$$

and

$$\sum_{n \le x} \left|[g_1(n+1)]^k - [g_1(n)]^k\right| = o(x) \quad \text{as } x \to \infty, \tag{55}$$

hold for each positive integer k, where in the last step we have used induction on k.

Let

$$\mathcal{G}(n) := [g_1(n)]^k \quad (n = 1, 2, \dots),$$

where k is a constant satisfying (15). It is obvious that $\mathcal{G} \in \mathcal{M}^*(1)$.

Let

$$A(x) := \sum_{n \le x} \mathcal{G}(n).$$

By the result of Elliott [2] it follows that for each sufficiently large real number w_0, there exists a real number $t(x)$ satisfying $|t(x)| \le (\log x)^{1/19}$ such that for $1 \le Q \le w_0$ and $x > w_0$ we have

$$A(x/Q) = \sum_{n \le x/Q} \mathcal{G}(n) = Q^{-1-it(x)} A(x) + O\left[\frac{x}{Q}\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right].$$

Let $Q \ge 2$ be an integer. From the above relation and the fact $\mathcal{G} \in \mathcal{M}^*(1)$, we have

$$\sum_{\substack{n \le x \\ n \equiv 0 \pmod{Q}}} \mathcal{G}(n) = \mathcal{G}(Q) \sum_{m \le x/Q} \mathcal{G}(m) \tag{56}$$

$$= Q^{-1-it(x)} \mathcal{G}(Q) A(x) + O\left[\frac{x}{Q}\mathcal{G}(Q)\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right].$$

Let R be an integer for which $0 \le R < Q$. Then, by (55) and (56), we deduce that

$$\sum_{\substack{n \le x \\ n \equiv R \pmod{Q}}} \mathcal{G}(n) = \sum_{Qm+R \le x} [\mathcal{G}(Qm+R) - \mathcal{G}(Qm)] + \sum_{Qm+R \le x} \mathcal{G}(Qm)$$

$$= Q^{-1-it(x)} \mathcal{G}(Q) A(x) + O\left[\frac{x}{Q}\mathcal{G}(Q)\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right] + o(x)$$

holds for each integer $0 \le R \le Q-1$. Thus, by adding the above relations, we get

$$A(x) = Q^{-it(x)} \mathcal{G}(Q) A(x) + O\left[x\mathcal{G}(Q)\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right] + o(Qx). \tag{57}$$

By (15), we can choose a constant $C > 0$ and a sequence $\{x_i\}_{i=1}^{\infty}$, $x_i \to \infty$ such that

$$\left|\frac{A(x_i)}{x_i}\right| \geq C > 0 \quad \text{as } x_i \to \infty.$$

Then (57) gives

$$C\left|1 - \frac{\mathcal{G}(Q)}{Q^{it(x_i)}}\right| \leq \left|1 - \frac{\mathcal{G}(Q)}{Q^{it(x_i)}}\right| . \left|\frac{A(x_i)}{x_i}\right| = o(1),$$

and so

$$Q^{it(x_i)} \to \mathcal{G}(Q) \quad \text{as } x_i \to \infty. \tag{58}$$

Since (58) holds for all integers Q for which $1 \leq Q \leq w_0$, and for each Q we get from (58) that

$$t(x_i) \to t \quad \text{as } x_i \to \infty, \tag{59}$$

thus (58) and (59) imply that

$$\mathcal{G}(Q) = Q^{it} \tag{60}$$

for all $1 \leq Q \leq w_0$. This with $w_0 \to \infty$ shows that (60) holds for all positive integers Q.

Since

$$\mathcal{G}(n) = [g_1(n)]^k \quad \text{and} \quad \mathcal{G}(n) = n^{it} \quad (n = 1, 2, \dots),$$

it follows that for each positive integer n there is a complex number $G(n)$ such that

$$g_1(n) = n^{it/k} G(n). \tag{61}$$

It is obvious that $G \in \mathcal{M}^*(1)$ and

$$[G(n)]^k = 1 \quad (n = 1, 2, \dots).$$

Let $\tau := t/k$. By (61) we have

$$G(n+1) - G(n) = \frac{g_1(n+1) - g_1(n)}{n^{i\tau}} - G(n+1)\frac{(n+1)^{i\tau} - (n)^{i\tau}}{n^{i\tau}},$$

which with (55) implies that

$$\sum_{n\le x} |G(n+1) - G(n)| = o(x) \quad \text{as } x \to \infty.$$

This completes the proof of Theorem 4.

Remark. We can use Theorem 2 of A. HILDEBRAND [5] to prove (60), i.e.

$$[g_1(Q)]^k = Q^{it} \quad (Q = 1, 2, \dots)$$

holds for some real number t. Indeed, by using Halász' theorem, it follows by (15) that for some real number t

$$\operatorname{Re} \sum_p \frac{1}{p} \left(1 - \frac{(g_1(p))^k}{p^{-it}}\right) < \infty,$$

the series being taken over all primes p. It follows from Theorem 2 of A. HILDEBRAND [5] that the above inequality implies

$$\frac{1}{x} \sum_{n\le x} \frac{(g_1(n))^k}{(g_1(n+1))^k} \to \prod_p F_p,$$

where

$$F_p = 1 - \frac{2}{p} + 2\left(1 - \frac{1}{p}\right) \operatorname{Re} \frac{(g_1(p))^k p^{-it}}{p - (g_1(p))^k p^{-it}}.$$

Thus, the last relations with (55) jointly imply that $F_p = 1$ holds for each prime p, i.e. $(g_1(p))^k = p^{it}$. Thus, by using the complete multiplicativity of g_1, we obtain (60).

References

[1] P. D. T. A. ELLIOTT, Probabilistic number theory I: Mean-value theorems, Grund. Math. Wiss. **239**, *Springer-Verlag, New York, Berlin*, 1979.

[2] P. D. T. A. ELLIOTT, Extrapolating the mean-values of multiplicative functions, *Indagationes Math.* **51** (1989), 409–420.

[3] A. HILDEBRAND, Multiplicative functions at consecutive integers I., *Math. Proc. Camb. Phil. Soc.* **100** (1986), 229–236.

[4] A. HILDEBRAND, Multiplicative functions at consecutive integers II., *Math. Proc. Camb. Phil. Soc.* **103** (1988), 389–398.

[5] A. Hildebrand, An Erdős–Wintner theorem for differences of additive functions, *Trans. Amer. Math. Soc.* **310** (1988), 257–276.

[6] I. Kátai, Some problems in number theory, *Studia Sci. Math. Hungar.* **16** (1981), 289–295.

[7] I. Kátai, Multiplicative functions with regularity properties II., *Acta Math. Hungar.* **43** (1984), 105–130.

[8] I. Kátai, Multiplicative functions with regularity properties VI., *Acta Math. Hungar.* **58** (1991), 343–350.

[9] J. L. Mauclaire and L. Murata, On the regularity of arithmetic multiplicative functions I., *Proc. Japan Acad. Ser. A. Math. Sci.* **56** (1980), 438–440.

[10] B. M. Phong, Note on a theorem of J. L. Mauclaire and Leo Murata on multiplicative functions, *Ann. Univ. Sci. Budapest. Eötvös, Sec. Math.* **33** (1990), 247–251.

[11] B. M. Phong, A characterization of some arithmetical multiplicative functions, *Acta Math. Hungar.* **63** (1) (1994), 29–43.

[12] B. M. Phong, On a theorem of Kátai–Wirsing, *Acta Sci. Math. Szeged* **56** (1992), 237–247.

[13] B. M. Phong, A note on multiplicative functions with regularity properties, *Publ. Math. Debrecen* **41** (1992), 117–125.

[14] B. M. Phong, Characterizations of the logarithm as an additive function, *Ann. Univ. Sci. Budapest. Eötvös, Sec. Comp.* **16** (1996), 45–67.

[15] A. Sárközy, On multiplicative arithmetic functions satisfying a linear recursion, *Studia. Sci. Math. Hungar.* **13** (1978), 79–104.

[16] E. Wirsing, Tang Yuansheng and Shao Pintsung, On a conjecture of Kátai for additive functions, *J. Number Theory* **56** (1996), 391–395.

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