

Symmetric words in free nilpotent groups of class 4

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Abstract. A word $w(X_1, \dots, X_n)$ is called n -symmetric for a given group G if $w(g_1, \dots, g_n) = w(g_{\sigma(1)}, \dots, g_{\sigma(n)})$ for all $g_1, \dots, g_n$ in G and all permutations σ from the symmetric group S_n . In this note we describe n -symmetric words in the free nilpotent groups of class 4.

1. Preliminaries and main results

The problem of characterizing the n -symmetric words in the given group G was initiated by PŁONKA [8]–[10] who gave a complete description of the n -symmetric words in nilpotent groups of class ≤ 3 . For results for metabelian and other groups we refer to [1]–[6].

Let F_n denote the absolutely free group on $X_1, \dots, X_n$.

Definition. A word $w(X_1, \dots, X_n) \in F_n$ is called n -symmetric word for a group G if $w(g_{\sigma(1)}, \dots, g_{\sigma(n)}) = w(g_1, \dots, g_n)$ for all $g_1, \dots, g_n \in G$ and all permutations σ from the symmetric group S_n .

It follows from the definition that we can restrict ourselves to relatively free groups with n free generators and to natural action of S_n on them. Let $F_n(G)$ be the relatively free group on $x_1, \dots, x_n$ in a variety generated by the group G . Let A be the group of automorphisms of $F_n(G)$ induced by the mappings $x_i \longrightarrow x_{\sigma(i)}$, $1 \leq i \leq n$, ($\sigma \in S_n$). The group

$$S^{(n)}(G) = \{w \in F_n(G) : w = \alpha(w) \text{ for every } \alpha \in A\}$$

is called a group of n -symmetric words for G .

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In this paper we describe $S^{(n)}(G)$ in the case of G , the free nilpotent group of class 4 which we denote shortly by $S^{(n)}(\mathfrak{N}_4)$ ($\mathfrak{N}_c$ – variety of nilpotent groups of class c). Our results extend these from [8], [10] and give a correction to one statement in [5].

We denote by $[x, y] = x^{-1}y^{-1}xy$ a commutator of elements x, y . Commutators of higher weight are defined as left-normed.

Let $u_1(x, y) = [y, x, x][y, x, y]^{-1}$, $u_2(x, y) = [y, x, x, x][y, x, y, y]^{-1}$.

Theorem 1. *The group $S^{(2)}(\mathfrak{N}_4)$ is a free nilpotent group of class 2 generated by $u_1(x, y)$, $u_2(x, y)$ and $u_3 = x^4y^4[y, x]^8[y, x, x]^{24}[y, x, x, x]^{16} \times [y, x, x, y]^{18}$.*

The Theorem 1 answers affirmatively a question raised in [9]. We note here that all groups $S^{(n)}(\mathfrak{N}_c)$ are abelian if $c \leq 3$.

Theorem 2. *The group $S^{(3)}(\mathfrak{N}_4)$ is a free abelian group generated by*
 $w_1(x, y, z) = u_1(x, y)u_1(x, z)u_1(y, z),$
 $w_2(x, y, z) = u_2(x, y)u_2(x, z)u_2(y, z).$

Theorem 3. *The group $S^{(4)}(\mathfrak{N}_4)$ is a free abelian group generated by*
 $w_3(x, y, z, t) = u_1(x, y)u_1(x, z)u_1(x, t)u_1(y, z)u_1(y, t)u_1(z, t),$
 $w_4(x, y, z, t) = u_2(x, y)u_2(x, z)u_2(x, t)u_2(y, z)u_2(y, t)u_2(z, t).$

Since we have isomorphisms $S^{(n)}(\mathfrak{N}_4) \cong S^{(4)}(\mathfrak{N}_4)$ (for $n > 4$ [9]), our theorems give a full description of n -symmetric words for any natural n .

A map $w(x_1, \dots, x_n, x_{n+1}) \rightarrow w(x_1, \dots, x_n, 1)$ induces homomorphism

$$\delta_n^{n+1}(\mathfrak{N}_c) : S^{(n+1)}(\mathfrak{N}_c) \rightarrow S^{(n)}(\mathfrak{N}_c).$$

It is clear that $\delta_n^{n+1}(\mathfrak{N}_4)$ is an isomorphism for $n \geq 3$. However, $\delta_2^3(\mathfrak{N}_4)$ is a monomorphism, which contradicts a second part of Theorem 3 from [5] which states that $\delta_n^{n+1}(\mathfrak{N}_{n+2})$ is not a monomorphism for any n . In fact, a sketch of the proof given in [5] shows that $\delta_n^{n+1}(\mathfrak{N}_{n+2})$ is not monomorphism for $n > 2$. This raise a question of checking the validity of this statement from [5] for other nilpotent groups of class 4.

2. Identities in nilpotent groups

We use a standard definitions from [7] without explanations.

We need some well known identities:

$$\begin{aligned} (1) \quad [x^{-1}, y] &= [x, y]^{-1}[y, x, x^{-1}], & (2) \quad [x, y^{-1}] &= [x, y]^{-1}[y, x, y^{-1}] \\ (3) \quad [xy, z] &= [x, z][x, z, y][y, z], & (4) \quad [x, yz] &= [x, z][x, y][x, y, z] \end{aligned}$$

valid in arbitrary groups. We use notation $\binom{n}{i} = \frac{1}{i!} \cdot n(n-1) \cdots (n-i+1)$.

Now we list identities of nilpotent groups of class 4 which we use in next sections to rewrite some words as the products of basic commutators. We fix a natural order of basic commutators:

$$x < y < z < t < [y, x] < [z, x] < [t, x] < [z, y] < [t, y] < [t, z] < \dots$$

Lemma 1. *The following identities hold in a nilpotent group G of class four for any $x, y, z, t \in G$ and all integers n, m, k, l .*

$$\begin{aligned} (5) \quad [y^n, x^m] &= [y, x]^{nm} [y, x, x]^{n\binom{m}{2}} [y, x, y]^{(n)_2^m} [y, x, x, x]^{n\binom{m}{3}} \\ &\quad \times [y, x, x, y]^{(n)_2^m} [y, x, y, y]^{(n)_3^m}, \\ (6) \quad [y^n, x^m, z^k] &= [y, x, z]^{nmk} [y, x, y, z]^{(n)_2^{mk}} \\ &\quad \times [y, x, x, z]^{(n)_2^{mk}} [y, x, z, z]^{(k)_2^{nm}}, \\ (7) \quad [y^n, x^m, z^k, t^l] &= [y, x, z, t]^{nmkl}. \end{aligned}$$

PROOF. Using (1)–(4) one can prove that

$$\begin{aligned} [y^{-1}, x] &= [y, x]^{-1}[y, x, y][y, x, y, y]^{-1}, \\ [y, x^{-1}] &= [y, x]^{-1}[y, x, x][y, x, x, x]^{-1} \end{aligned}$$

and by induction the following identities for all natural n, m

$$\begin{aligned} [y^n, x] &= [y, x]^n [y, x, y]^{(n)_2} [y, x, y, y]^{(n)_3}, \\ [y, x^m] &= [y, x]^m [y, x, x]^{(m)_2} [y, x, x, x]^{(m)_3}. \end{aligned}$$

Now we have $[y, x^{-m}] = [y, (x^m)^{-1}] = [y, x^m]^{-1} [y, x^m, x^m] \times [y, x^m, x^m, x^m]^{-1} = [y, x]^{-m} [y, x, x]^{\binom{-m}{2}} [y, x, x, x]^{\binom{-m}{3}}$ so, this identity is valid for all integers. Similarly we obtain the expression for $[y^{-n}, x]$. Finally, for all integers n, m , we have

$$[y^n, x^m] = [y^n, x]^m [y^n, x, x]^{\binom{m}{2}} [y^n, x, x, x]^{\binom{m}{3}} = \prod_{i,j>0}^{i+j<5} [y, x, x_{(j-1)} y]^{\binom{n}{i} \binom{m}{j}}.$$

Using this identity one can easily prove (6); (7) is easy to check directly. $\square$

Lemma 2. *The following identities hold in any nilpotent group of class four:*

- (8) $[x, y, z] = [y, x, z]^{-1},$
- (9) $[z, y, x] = [z, x, y][y, x, z]^{-1}[[z, x], [y, x]][[z, y], [y, x]][[z, y], [z, x]],$
- (10) $[x, y, z, t] = [y, x, z, t]^{-1},$
- (11) $[y, x, t, z] = [y, x, z, t][[z, t], [y, x]],$
- (12) $[z, y, x, t] = [z, x, y, t][y, x, z, t]^{-1},$
- (13) $[t, y, x, z] = [t, x, y, z][y, x, z, t]^{-1}[[t, z], [y, x]],$
- (14) $[z, y, t, x] = [z, x, y, t][y, x, z, t]^{-1}[[t, x], [z, y]]^{-1},$
- (15) $[t, y, z, x] = [t, x, y, z][y, x, z, t]^{-1}[[t, y], [z, x]][[t, z], [y, x]].$

PROOF. (8) and (10) follow easily from (1)–(4). (9) is the Jacobi identity. We have

$$\begin{aligned} [xy, zt] &= [xy, t][xy, z][xy, z, t] \\ &= [x, t][x, t, y][y, t][x, z][x, z, y][y, z][x, z, t][x, z, y, t][y, z, t] \end{aligned}$$

and similarly

$$\begin{aligned} [xy, zt] &= [x, zt][x, zt, y][y, zt] \\ &= [x, t][x, z][y, t][y, z][x, z, t][y, z, t][x, t, y][x, z, y][x, z, t, y] \end{aligned}$$

which implies (11). By Jacobi identity we have

$$\begin{aligned}
 [z, y, x, t] &= [z, y, x]^{-1} t^{-1} [z, y, x] t = [y, x, z] [z, x, y]^{-1} [[z, y], [z, x]]^{-1} \\
 &\quad \times [[z, y], [y, x]]^{-1} [[z, x], [y, x]]^{-1} t^{-1} [z, x, y] [y, x, z]^{-1} \\
 &\quad \times [[z, y], [z, x]] [[z, y], [y, x]] [[z, x], [y, x]] t \\
 &= [y, x, z] [z, x, y]^{-1} t^{-1} [z, x, y] t [y, x, z]^{-1} [y, x, z, t]^{-1} \\
 &= [y, x, z] [z, x, y, t] [y, x, z]^{-1} [y, x, z, t]^{-1}
 \end{aligned}$$

which gives us (12). (13) follows from

$$[t, y, x, z] \stackrel{(11)}{=} [t, x, y, z] [y, x, t, z]^{-1} \stackrel{(12)}{=} [t, x, y, z] [y, x, z, t]^{-1} [[z, t], [y, x]]^{-1}.$$

Similarly we have

$$[z, y, t, x] \stackrel{(11)}{=} [z, y, x, t] [[t, x], [z, y]]^{-1} \stackrel{(12)}{=} [z, x, y, t] [y, x, z, t]^{-1} [[t, x], [z, y]]^{-1}$$

and

$$\begin{aligned}
 [t, y, z, x] &\stackrel{(11)}{=} [t, y, x, z] [[t, y], [z, x]] \stackrel{(12)}{=} [t, x, y, z] [y, x, t, z]^{-1} [[t, y], [z, x]] \\
 &\stackrel{(11)}{=} [t, x, y, z] [y, x, z, t]^{-1} [[t, z], [y, x]] [[t, y], [z, x]]. \quad \square
 \end{aligned}$$

We need a characterization of elements of $S^{(2)}(\mathfrak{N}_4)$. Every element from $S^{(n)}(\mathfrak{N}_4)$ has a form $x_1^a x_2^a \dots x_n^a \cdot c$, where c belongs to the commutator subgroup (see Lemma 4 of [2]). Moreover, we have

Lemma 3. *An element $w(x, y)$ from $F_2(\mathfrak{N}_4)$ belongs to $S^{(2)}(\mathfrak{N}_4)$ if and only if*

$$w(x, y) = x^a y^a [y, x]^b [y, x, x]^{c_1} [y, x, y]^{c_2} [y, x, x, x]^{d_1} [y, x, x, y]^{d_2} [y, x, y, y]^{d_3}$$

where

$$a^2 = 2b, \quad c_1 + c_2 = a \binom{a}{2}, \quad d_1 + d_3 = a \binom{a}{3}, \quad 2d_2 = \binom{a}{2} \binom{a}{2}.$$

PROOF. We have to prove the equality

$$\begin{aligned}
w(y, x) &= y^a x^a [x, y]^b [x, y, y]^{c_1} [x, y, x]^{c_2} [x, y, y, y]^{d_1} [x, y, y, x]^{d_2} [x, y, x, x]^{d_3} \\
&= x^a y^a [y^a, x^a] [y, x]^{-b} [y, x, x]^{-c_2} [y, x, y]^{-c_1} [y, x, x, x]^{-d_3} \\
&\quad \times [y, x, x, y]^{-d_2} [y, x, y, y]^{-d_1} \\
&= x^a y^a [y, x]^{a^2-b} [y, x, x]^{a \cdot \binom{a}{2} - c_2} [y, x, y]^{a \cdot \binom{a}{2} - c_1} [y, x, x, x]^{a \cdot \binom{a}{3} - d_3} \\
&\quad \times [y, x, x, y]^{\binom{a}{2} \binom{a}{2} - d_2} [y, x, y, y]^{a \cdot \binom{a}{3} - d_1} = w(x, y).
\end{aligned}$$

The lemma now follows from the fact that in the free nilpotent group a presentation of the element as a product of basic commutators is unique [7]. $\square$

3. Proofs of main results

Now we are ready to prove our theorems.

PROOF of Theorem 1. It follows from the Lemma 3 that every element of $S^{(2)}(\mathfrak{N}_4)$ has a form

$$\begin{aligned}
&x^{4m} y^{4m} [y, x]^{8m^2} [y, x, x]^c [y, x, y]^{8m^2(4m-1)-c} [y, x, x, x]^d \\
&\quad \times [y, x, x, y]^{2m^2(4m-1)^2} [y, x, y, y]^{\frac{1}{3}8m^2(4m-1)(4m-2)-d},
\end{aligned}$$

where m, c, d are arbitrary integers. So, the group $S^{(2)}(G)$ is generated by three elements

$$\begin{aligned}
u_1 &= [y, x, x][y, x, y]^{-1}, & u_2 &= [y, x, x, x][y, x, y, y]^{-1}, \\
u_3 &= x^4 y^4 [y, x]^8 [y, x, x]^{24} [y, x, x, x]^{16} [y, x, x, y]^{18}.
\end{aligned}$$

We have $u_3 u_1 \neq u_1 u_3 = u_3 u_1 u_2^4$ and commutator of any two 2-symmetric words from $S^{(2)}(\mathfrak{N}_4)$ belongs to the centre, so the theorem is proved. $\square$

PROOF of Theorem 2. Every element of $S^{(3)}(\mathfrak{N}_4)$ has a form

$$v(x, y, z) = x^a y^a z^a [y, x]^b [z, x]^b [z, y]^b v_1(x, y) v_2(x, z) v_3(y, z) v_0(x, y, z)$$

where

$$v_i(x, y) = [y, x, x]^{c_{i,1}} [y, x, y]^{c_{i,2}} [y, x, x, x]^{d_{i,1}} [y, x, x, y]^{d_{i,2}} [y, x, y, y]^{d_{i,3}}$$

and v_0 is a product of basic commutators on exactly three letters. Simple calculation using transpositions of generators, shows that $v_1 = v_2 = v_3$. Since $v(x, y, 1)$ belongs to $S^{(2)}(\mathfrak{N}_4)$, we can apply Lemma 3. So we have

$$v_1(x, y) = [y, x, x]^{c_1} [y, x, y]^{c_2} [y, x, x, x]^{d_1} [y, x, x, y]^{d_2} [y, x, y, y]^{d_3}$$

and $a, b, c_1, c_2, d_1, d_2, d_3$ satisfy the conditions of Lemma 3. We put

$$\begin{aligned} v_0(x, y, z) &= [y, x, z]^{c_3} [z, x, y]^{c_4} [y, x, x, z]^{d_4} [y, x, y, z]^{d_5} [y, x, z, z]^{d_6} \\ &\quad \times [z, x, x, y]^{d_7} [z, x, y, y]^{d_8} [z, x, y, z]^{d_9} \\ &\quad \times [[z, x], [y, x]]^{e_1} [[z, y], [y, x]]^{e_2} [[z, y], [z, x]]^{e_3} \end{aligned}$$

and rewrite the element $v(y, x, z)$ as a product of basic commutators. We consider now only the basic commutators on three letters. By rewriting v_0 we obtain

$$\begin{aligned} v_0(y, x, z) &= [y, x, z]^{-c_3-c_4} [z, x, y]^{c_4} [y, x, x, z]^{-d_5-d_8} [y, x, y, z]^{-d_4-d_7} \\ &\quad \times [y, x, z, z]^{-d_6-d_9} [z, x, x, y]^{d_8} [z, x, y, y]^{d_7} [z, x, y, z]^{d_9} \\ &\quad \times [[z, x], [y, x]]^{-e_2+c_4+2d_8} [[z, y], [y, x]]^{-e_1+c_4+2d_7} \\ &\quad \times [[z, y], [z, x]]^{-e_3+c_4} \end{aligned}$$

and from $v(y, x, z)(v_0(y, x, z))^{-1}$ we have

$$[y, x, z]^{a^3} [y, x, x, z]^{a^2 \binom{a}{2}} [y, x, y, z]^{a^2 \binom{a}{2}} [y, x, z, z]^{a^2 \binom{a}{2}} [[z, y], [z, x]]^{b^2}.$$

The same calculation for $v(y, z, x)$ gives us

$$\begin{aligned} v_0(y, z, x) &= [y, x, z]^{-c_3-c_4} [z, x, y]^{c_3} [y, x, x, z]^{-d_6-d_9} [y, x, y, z]^{-d_4-d_7} \\ &\quad \times [y, x, z, z]^{-d_5-d_8} [z, x, x, y]^{d_6} [z, x, y, y]^{d_4} [z, x, y, z]^{d_5} \\ &\quad \times [[z, x], [y, x]]^{e_3+c_3-d_9+2d_6} [[z, y], [y, x]]^{e_1+c_3+2d_4} \\ &\quad \times [[z, y], [z, x]]^{e_2+c_3+d_5} \end{aligned}$$

from rewriting $v_0(y, z, x)$ and from $v(y, z, x)(v_0(y, z, x))^{-1}$

$$\begin{aligned} &[y, x, z]^{a^3} [y, x, x, z]^{a^2 \binom{a}{2}} [y, x, y, z]^{a^2 \binom{a}{2}} [y, x, z, z]^{a^2 \binom{a}{2}} \\ &\quad \times [[z, x], [y, x]]^{-b^2} [[z, y], [y, x]]^{-b^2} [[z, y], [z, x]]^{-b^2}. \end{aligned}$$

Comparing the powers of basic commutators we obtain

$$a^2 = 2b, \quad c_3 = c_4 = c, \quad d_4 = d_6 = d_7 = d_8 = d, \quad d_5 = d_9 = d_0, \quad 3c = a^3,$$

$$2d + d_0 = a^2 \binom{a}{2}, \quad 2d_0 = b^2, \quad e_1 + e_2 = c + 2d, \quad 2e_3 = c + b^2,$$

$$e_1 + b^2 + d_0 = e_3 + c + 2d, \quad e_2 + b^2 = e_1 + c + 2d, \quad e_3 + b^2 = e_2 + c + d_0.$$

This implies $3c + 4d - 6d_0 = 0$ and for some integer k the equalities $a = 6k$, $b = 18k^2$, $c = 12k^2$, $d_0 = 3^4 \cdot 2k^4$ and $d = 2 \cdot 3^3 k^3 (6k - 1) - 3^4 \cdot k^4$. But then we obtain $k^2(1 - 6k) = 0$ and consequently $k = 0$ and $e_1 = e_2 = e_3 = 0$, which finishes the proof. $\square$

PROOF of Theorem 3. Let $w = w(x, y, z, t)$ belong to $S^{(4)}(\mathfrak{N}_4)$ and let

$$w_2 = x^a y^a z^a t^a [y, x]^b [z, x]^b [t, x]^b [z, y]^b [t, y]^b [t, z]^b.$$

Since the words $w(x, y, z, 1)$, $w(x, y, 1, t)$, $w(x, 1, z, t)$, $w(1, y, z, t)$ are both in $S^{(3)}(\mathfrak{N}_4)$ we have

$$w(x, y, z, t) = w_2 w_1(x, y, z) w_1(x, y, t) w_1(x, z, t) w_1(y, z, t) w_0 = w'_1 \cdot w_0,$$

where w'_1 is a product of commutators which contain exactly 3 letters and

$$\begin{aligned} w_0 &= [y, x, z, t]^{f_1} [z, x, y, t]^{f_2} [t, x, y, z]^{f_3} [[t, x], [z, y]]^{f_4} \\ &\quad \times [[t, y], [z, x]]^{f_5} [[t, z], [y, x]]^{f_6} \end{aligned}$$

is the product of all basic commutators on exactly 4 letters and w_2 is trivial because $a = b = 0$.

Using Lemmas 1, 2 we rewrite $w(y, x, z, t)$ as a product of basic commutators. Then we obtain

$$\begin{aligned} &[y, x, z, t]^{-f_1 - f_2 - f_3} [z, x, y, t]^{f_2} [t, x, y, z]^{f_3} [[t, x], [z, y]]^{f_5} \\ &\quad \times [[t, y], [z, x]]^{f_4} [[t, z], [y, x]]^{-f_6}. \end{aligned}$$

So we deduce that

$$2f_1 + f_2 + f_3 = 0, \quad f_4 = f_5, \quad f_5 = f_4, \quad 2f_6 = f_3.$$

The similar calculations for the element $w(y, z, t, x)$ give

$$[y, x, z, t]^{-f_1-f_2-f_3} [z, x, y, t]^{f_1} [t, x, y, z]^{f_2} [[t, x], [z, y]]^{-f_1-f_6} \\ \times [[t, y], [z, x]]^{f_2+f_5} [[t, z], [y, x]]^{f_2+f_4}.$$

It follows that $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0$ and Theorem 3 is proved. $\square$

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