

The homogeneous lift to the tangent bundle of a Finsler metric

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Abstract. Notice that the Sasaki–Matsumoto lift $\overset{0}{\mathbb{G}}$ from (1.4) of a Finsler metric tensor g is not homogeneous on the fibers of the tangent bundle. We correct this inconvenient by introducing a new kind of lift $\mathbb{G}$ of g , given by (2.1), which is 0-homogeneous. Some properties of the Riemannian space $(\widetilde{TM}, \mathbb{G})$ are studied. The almost complex structure $\mathbb{F}$, from (3.1) is introduced. It has the property of homogeneity and $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure. We prove that in fact $(\mathbb{G}, \mathbb{F})$ is a conformal almost Kählerian structure. It depends only on the fundamental function of the Finsler space considered.

Introduction

The Sasaki–Matsumoto lift $\overset{0}{\mathbb{G}}$ [6], [11] to the manifold $\widetilde{TM} = TM \setminus \{O\}$ of a Finsler metric tensor g is extremely important in the study of the geometry of a Finsler space $F^n = (M, F(x, y))$. $\overset{0}{\mathbb{G}}$ determines a Riemannian structure on $\widetilde{TM}$, which depends only on the fundamental function F . It is not difficult to see that $\overset{0}{\mathbb{G}}$ does not have a Finslerian meaning. More precisely, $\overset{0}{\mathbb{G}}$ is not homogeneous with respect to the vertical variables y^i . Consequently, we cannot study global properties – as the Gauss–Bonnet Theorem – for the Finsler space F^n by means of this lift [4], [5]. Also, since the two terms of the metric $\overset{0}{\mathbb{G}}$ do not have the same physical dimensions, it does not satisfy the principles of the Post-Newtonian Calculus and so it is not convenient for a gauge theory.

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In the present paper, using the same ideas as in the Riemannian case [10] we define a new lift $\mathbb{G}$, (2.1), to $\widetilde{TM}$, which depends only on the fundamental function F of the Finsler space F^n and which is 0-homogeneous on the fibers of the tangent bundle TM .

Some geometrical properties of the space $(\widetilde{TM}, \mathbb{G})$ are studied by means of the Cartan nonlinear connection of the space F^n : the canonical metrical N -connection, the Levi-Civita connection of $\mathbb{G}$ etc.

We introduce the natural almost complex structure $\mathbb{F}$ by the formulae (3.1). It has the property of homogeneity and depends only on F . The main result is as follows: The space $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$ is almost Hermitian and its associated almost symplectic structure θ is such that $d\theta = 0$ (modulo θ). We prove that this space is in fact conformal almost Kählerian. It represents the geometrical model of the Finsler space F^n with respect to the homogeneous lift $\mathbb{G}$.

1. Preliminaries

Let $F^n = (M, F)$ be a Finsler space, M being a real n -dimensional differentiable manifold and $F : TM \rightarrow \mathbb{R}$ its fundamental function. F is of C^∞ -class on $\widetilde{TM} = TM \setminus \{O\}$ and continuous on the null section of the natural projection $\pi : TM \rightarrow M$. The fundamental tensor field of F^n is

$$(1.1) \quad g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad \forall (x, y) \in \widetilde{TM}.$$

The regular Lagrangian $F^2(x, y)$ determines the canonical spray $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ with the coefficients $G^i = \frac{1}{2} \gamma_{jk}^i(x, y) y^j y^k$, where $\gamma_{jk}^i(x, y)$ are the Christoffel symbols of the metric tensor $g_{ij}(x, y)$. The Cartan nonlinear connection N of the space F^n has the coefficients $N_j^i = \frac{\partial G^i}{\partial y^j}$.

N determines a distribution on $\widetilde{TM}$, which is supplementary to the vertical distribution V . We have the following direct sum of linear spaces:

$$(1.2) \quad T_u(\widetilde{TM}) = N_u \oplus V_u, \quad \forall u \in \widetilde{TM}.$$

An adapted basis to N_u and V_u is given by $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, $(i = 1, \dots, n)$, where

$$(1.3) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y) \frac{\partial}{\partial y^j}.$$

The dual basis is $(dx^i, \delta y^i)$ with

$$(1.3) \quad \delta y^i = dy^i + N_j^i(x, y) dx^j.$$

M. MATSUMOTO [6] extended to Finsler spaces F^n the notion of Sasaki lift, considering the tensor field

$$(1.4) \quad \overset{0}{\mathbb{G}}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j, \quad \forall (x, y) \in \widetilde{TM}.$$

It easily follows that $\overset{0}{\mathbb{G}}$ is a Riemannian metric globally defined on $\widetilde{TM}$ and depending only on the fundamental function F of the Finsler space F^n .

Let us consider the homothety $h_t : (x, y) \rightarrow (x, ty)$, $t \in \mathbb{R}^*$ on the fibers of the tangent bundle TM . Then $\overset{0}{\mathbb{G}}$ is transformed as follows:

$$\overset{0}{\mathbb{G}} \circ h_t(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + t^2 g_{ij}(x, y) \delta y^i \otimes \delta y^j, \quad \forall t \in \mathbb{R}^*.$$

We see that the Sasaki–Matsumoto lift $\overset{0}{\mathbb{G}}$ is not homogeneous on the fibers of the tangent bundle TM .

Next we consider the $\mathcal{F}(\widetilde{TM})$ -linear mapping $\mathbb{F} : \chi(\widetilde{TM}) \rightarrow \chi(\widetilde{TM})$, defined by

$$(1.5) \quad \overset{0}{\mathbb{F}} \left(\frac{\delta}{\delta x^i} \right) = -\frac{\partial}{\partial y^i}, \quad \overset{0}{\mathbb{F}} \left(\frac{\partial}{\partial y^i} \right) = \frac{\delta}{\delta x^i}, \quad (i = 1, \dots, n).$$

As $\overset{0}{\mathbb{F}}$ maps the 1-homogeneous vector field $\frac{\delta}{\delta x^i}$ onto 0-homogeneous vector fields $\frac{\partial}{\partial y^i}$, $(i = 1, \dots, n)$, it does not preserve the property of homogeneity of the vector fields on $\widetilde{TM}$.

It is known that $\overset{0}{\mathbb{F}}$ is an almost complex structure on $\widetilde{TM}$ depending only on the fundamental function F which becomes a complex structure on $\widetilde{TM}$ if and only if the horizontal distribution N is integrable.

Now let us consider the Cartan–Poincaré forms

$$(1.6) \quad \overset{\circ}{\omega} = \frac{1}{2} \frac{\partial F^2}{\partial y^i} dx^i, \quad \overset{\circ}{\theta} = g_{ij}(x, y) \delta y^i \wedge dx^j.$$

Evidently, $\overset{\circ}{\omega}$ and $\overset{\circ}{\theta}$ are globally defined on $\widetilde{TM}$ and $\overset{\circ}{\theta}$ is an almost symplectic structure on $\widetilde{TM}$.

As is known, between $\overset{\circ}{\omega}$ and $\overset{\circ}{\theta}$ there is the relation

$$(1.7) \quad d\overset{\circ}{\omega} = \overset{\circ}{\theta},$$

d being the exterior differential operator.

It follows that $\overset{\circ}{\theta}$ is a closed 2-form. In other words, $\overset{\circ}{\theta}$ is a symplectic structure. Remarking that the pair $(\overset{0}{\mathbb{G}}, \overset{0}{\mathbb{F}})$ is an almost Hermitian structure having $\overset{\circ}{\theta}$ as its associated symplectic structure, we recall the known result that $H^{2n} = (\widetilde{TM}, \overset{0}{\mathbb{G}}, \overset{0}{\mathbb{F}})$ is an almost Kählerian space.

In the terminology of the book [7], H^{2n} is the almost Kählerian model on $\widetilde{TM}$ of the Finsler space F^n considered. This is important for the geometry of the Finsler space F^n .

2. The homogeneous lift to $\widetilde{TM}$ of a Finsler metric

We define a new lift $\mathbb{G}$ on $\widetilde{TM}$ of the fundamental tensor field g_{ij} of a Finsler space $F^n = (M, F)$ which satisfies the following conditions:

- 1° $\mathbb{G}$ is 0-homogeneous with respect to y^i ;
- 2° It depends only on the fundamental function F ;
- 3° In the mechanical interpretation the terms of $\mathbb{G}$ have the same physical dimensions.

Definition 2.1. By the homogeneous lift to $\widetilde{TM}$ of the fundamental tensor field g_{ij} of a Finsler space F^n we mean the following tensor field on $\widetilde{TM}$:

$$(2.1) \quad \mathbb{G}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + \frac{a^2}{\|y\|^2} g_{ij}(x, y)\delta y^i \otimes \delta y^j,$$

$$\forall (x, y) \in \widetilde{TM},$$

where $a > 0$ is a constant and

$$(2.2) \quad \|y\|^2 = g_{ij}(x, y)y^i y^j = F^2(x, y).$$

The constant a is required by the applications, in order that the physical dimension of the terms of G be the same.

We get without difficulty the

Theorem 2.1. *The pair $(\widetilde{TM}, \mathbb{G})$ is a Riemannian space. $\mathbb{G}$ is 0-homogeneous on the fibers of TM and it depends only on the fundamental function $F(x, y)$ of the Finsler space F^n .*

We consider $\mathbb{G}$ as a (h, v) -metric, that is,

$$(2.3) \quad \mathbb{G} = \mathbb{G}^H + \mathbb{G}^V, \quad \mathbb{G}^H = g_{ij}(x, y) dx^i \otimes dx^j, \quad \mathbb{G}^V = h_{ij}(x, y) \delta y^i \otimes \delta y^j$$

$$(2.4) \quad h_{ij} = \frac{a^2}{\|y\|^2} g_{ij}(x, y).$$

Consequently, we can apply the theory of the (h, v) -Riemannian metric on $\widetilde{TM}$ investigated by R. MIRON and M. ANASTASIEI in the books [7] and [8].

The equation $F(x_0, y) = a$ determines the so called indicatrix of the Finsler space F^n in the point $x_0 \in M$, [6].

Therefore, we have the

Proposition 2.1. *The homogeneous lift $\mathbb{G}$ of the metric tensor $g_{ij}(x, y)$ coincides with the Sasaki–Matsumoto lift of $g_{ij}(x, y)$ on the indicatrix $F(x_0, y) = a$ for every point $x_0 \in M$.*

A linear connection D on $\widetilde{TM}$ is called a metrical N -connection with respect to $\mathbb{G}$, if $D\mathbb{G} = 0$ and D preserves by parallelism the horizontal distribution N .

As we know [7], [8], there exist the metrical N -connections on $\widetilde{TM}$. We represent a linear connection D on $\widetilde{TM}$ in the adapted basis in the following form:

$$(2.5) \quad \begin{aligned} D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} &= L_{jk}^H \frac{\delta}{\delta x^i} + \widetilde{L}_{jk}^i \frac{\partial}{\partial y^i}, & D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} &= \widetilde{\widetilde{L}}_{jk}^i \frac{\delta}{\delta x^i} + L_{jk}^V \frac{\partial}{\partial y^i}, \\ D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} &= C_{jk}^H \frac{\delta}{\delta x^i} + \widetilde{C}_{jk}^i \frac{\partial}{\partial y^i}, & D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} &= \widetilde{\widetilde{C}}_{jk}^i \frac{\delta}{\delta x^i} + C_{jk}^V \frac{\partial}{\partial y^i}. \end{aligned}$$

where $(L_{jk}^H, \widetilde{L}_{jk}^i, \widetilde{\widetilde{L}}_{jk}^i, L_{jk}^V, C_{jk}^H, \widetilde{C}_{jk}^i, \widetilde{\widetilde{C}}_{jk}^i, C_{jk}^V)$ are the coefficients of D .

By a direct calculation we obtain

Theorem 2.2. *There exist the metrical N -connections D on $\widetilde{TM}$ with respect to $\mathbb{G}$, which depend only on the fundamental function $F(x, y)$ of the Finsler space F^n . One of them has the following coefficients:*

$$(2.6) \quad \left\{ \begin{array}{l} \widetilde{L}_{jk}^i = \widetilde{\widetilde{L}}_{jk}^i = \widetilde{C}_{jk}^i = \widetilde{\widetilde{C}}_{jk}^i = 0 \\ \widetilde{L}_{jk}^H = \frac{1}{2} g^{is} \left(\frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} \right) \\ \widetilde{L}_{jk}^V = \frac{1}{2} h^{is} \left(\frac{\delta h_{sk}}{\delta x^j} + \frac{\delta h_{js}}{\delta x^k} - \frac{\delta h_{jk}}{\delta x^s} \right) \\ \widetilde{C}_{jk}^H = \frac{1}{2} g^{is} \left(\frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s} \right) \\ \widetilde{C}_{jk}^V = \frac{1}{2} h^{is} \left(\frac{\partial h_{sk}}{\partial y^j} + \frac{\partial h_{js}}{\partial y^k} - \frac{\partial h_{jk}}{\partial y^s} \right). \end{array} \right.$$

Of course, the structure equations of the previous connection can be written as in the books [7], [8].

For us it is important to express the coefficients $\left(\widetilde{L}_{jk}^H, \widetilde{L}_{jk}^i, \widetilde{\widetilde{L}}_{jk}^i, \widetilde{L}_{jk}^V, \widetilde{C}_{jk}^H, \widetilde{C}_{jk}^i, \widetilde{\widetilde{C}}_{jk}^i, \widetilde{C}_{jk}^V \right)$ of the Levi-Civita connection of the metric $\mathbb{G}$.

To this aim, expressing in the adapted basis the conditions

$$X\mathbb{G}(Y, Z) - \mathbb{G}(D_X Y, Z) - \mathbb{G}(Y, D_X Z) = 0$$

$$D_X Y - D_Y X - [X, Y] = 0$$

and using the torsions R_{jk}^i and P_{jk}^i of the Cartan connection $CT(N)$, we get:

Theorem 2.3. *The Levi-Civita connection of the Riemannian metric $\mathbb{G}$, in the adapted basis, has the following coefficients:*

$$(2.7) \quad \left\{ \begin{array}{l} \widetilde{L}_{jk}^H, \widetilde{C}_{jk}^V, \widetilde{C}_{jk}^H = \widetilde{\widetilde{L}}_{jk}^i = \widetilde{C}_{jk}^i + \frac{1}{2} g^{is} h_{mj} R_{sk}^m, \\ \widetilde{L}_{jk}^V = F_{jk}^i - \frac{1}{2} (\delta_s^i \delta_j^m - g_{sj} g^{im}) P_{km}^s, \quad \widetilde{C}_{jk}^i = \widetilde{L}_{jk}^i - B_{jk}^i, \\ \widetilde{L}_{jk}^i = -h^{is} C_{skj} + \frac{1}{2} R_{jk}^i, \quad \widetilde{\widetilde{C}}_{jk}^i = -h_{js} g^{im} \widetilde{C}_{mk}^s \end{array} \right.$$

where L_{jk}^H, C_{jk}^V are from (2.6) and (F_{jk}^i, C_{jk}^i) are the coefficients of the Cartan metrical connection of the Finsler space F^n .

Using the previous Levi-Civita connection we can study the main geometrical properties of the space $(\widetilde{TM}, \mathbb{G})$.

3. The almost Hermitian structure $(\mathbb{G}, \mathbb{F})$

The almost complex structure $\overset{0}{\mathbb{F}}$ defined by (1.5) does not preserve the property of homogeneity of the vector fields. It applies the 1-homogeneous vector fields $\frac{\delta}{\delta x^i}$, $(i = 1, \dots, n)$ onto the 0-homogeneous vector fields $\frac{\partial}{\partial y^i}$, $(i = 1, \dots, n)$.

We can eliminate this inconvenient by defining a new kind of almost complex structure $\mathbb{F} : \chi(\widetilde{TM}) \rightarrow \chi(\widetilde{TM})$, setting

$$(3.1) \quad \mathbb{F} \left(\frac{\delta}{\delta x^i} \right) = -\frac{\|y\|}{a} \frac{\partial}{\partial y^i}, \quad \mathbb{F} \left(\frac{\partial}{\partial y^i} \right) = \frac{a}{\|y\|} \frac{\delta}{\delta x^i}, \quad (i = 1, \dots, n).$$

Taking into account that the norm of the Liouville vector field, $\|y\|$, and the Cartan nonlinear connection N are defined on $\widetilde{TM}$, it is not difficult to prove

Theorem 3.1. *The following properties hold:*

- 1° $\mathbb{F}$ is a tensor field of type (1.1) on $\widetilde{TM}$.
- 2° $\mathbb{F} \circ \mathbb{F} = -I$.
- 3° $\mathbb{F}$ depends only on the fundamental function F of the Finsler space F^n .
- 4° The $\mathcal{F}(\widetilde{TM})$ -linear mapping $\mathbb{F} : \chi(\widetilde{TM}) \rightarrow \chi(\widetilde{TM})$ preserves the property of homogeneity of the vector fields from $\chi(\widetilde{TM})$.

It is important to know when is $\mathbb{F}$ a complex structure.

Theorem 3.2. $\mathbb{F}$ is a complex structure on $\widetilde{TM}$ if and only if the Finsler space F^n has the following property:

$$(3.2) \quad R_{ij}^h = \frac{1}{a^2} (y_i \delta_j^h - y_j \delta_i^h).$$

PROOF. The Nijenhuis tensor $\mathcal{N}_{\mathbb{F}}$:

$$\mathcal{N}_{\mathbb{F}}(X, Y) = \mathbb{F}^2[X, Y] + [\mathbb{F}X, \mathbb{F}Y] - \mathbb{F}[\mathbb{F}X, Y] - \mathbb{F}[X, \mathbb{F}Y]$$

vanishes if and only if the previous equations hold. $\square$

Remark. If F^n is reducible to a Riemannian space, then the equation (3.2) says that it is of constant sectional curvature.

The pair $(\mathbb{G}, \mathbb{F})$ has remarkable properties:

Theorem 3.3. *We have:*

- 1° $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure on $\widetilde{TM}$ and depends only on the fundamental function F of the Finsler space F^n .
- 2° The associated almost symplectic structure θ has the expression

$$(3.3) \quad \theta = \frac{a}{\|y\|} \overset{\circ}{\theta}$$

where $\overset{\circ}{\theta}$ is the symplectic structure (1.6).

- 3° The following formula holds:

$$(3.3)' \quad d\theta = -\frac{a}{\|y\|} d\|y\| \wedge \theta.$$

- 4° Consequently, $(\mathbb{G}, \mathbb{F})$ is a conformal almost Kählerian structure and we have

$$d\theta = 0 \text{ (modulo } \theta \text{)}.$$

The conformal almost Kählerian space $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$ is the geometrical model of the Finsler space F^n with respect to the homogeneous lift $\mathbb{G}$.

The previous considerations are important for the study of Finslerian gauge theory, [1]–[3], and in general in the Geometry of the Finsler space F^n .

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