

An existence theorem for the commutative neutrix product of distributions

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Abstract. In this paper we prove that the commutative neutrix product of the distributions x_+^{-r} and x_+^{-s} exists for $r, s = 1, 2, \dots$

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The distribution x_+^{-r} is defined by the equation

$$x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \dots$ and not as in GEL'FAND and SHILOV [6]. If we denote GEL'FAND and SHILOV's definition of x_+^{-r} by $F(x_+, -r)$, it was proved in [4] that

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r \phi(r-1)}{(r-1)!} \delta^{(r-1)}(x)$$

for $r = 1, 2, \dots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

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Our definition of x_+^{-r} is more convenient to use because it satisfies the equation

$$(x_+^{-r})' = -rx_+^{-r-1}$$

for $r = 1, 2, \dots$

Further, the distribution $x_+^{-1} \ln x_+$ is defined by

$$x_+^{-1} \ln x_+ = \frac{1}{2}(\ln^2 x_+)'$$

and in general, the distribution $x_+^{-r} \ln x_+$ is defined inductively by the equation

$$x_+^{-r} \ln x_+ = \frac{x_+^{-r} - (x_+^{-r+1} \ln x_+)'}{r-1}$$

for $r = 2, 3, \dots$

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$,
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition for the commutative neutrix product of two distributions was given in [3].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ and let $f_n(x) = (f * \delta_n)(x)$, $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \square g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\text{N-}\lim_{n \rightarrow \infty} \langle f_n(x)g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a, b) , where N is the neutrix, see van der CORPUT [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity. Further, if

$$\lim_{n \rightarrow \infty} \langle f_n(x) g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

we simply say that the product $f.g$ exists and equals h , see [2].

Before proving our main result, we note the following lemmas which are easily proved by induction.

Lemma 1. *If φ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$, then*

$$(1) \quad \langle x_+^{-r}, \varphi(x) \rangle = \int_0^1 x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!} \varphi^{(i)}(0) \right] dx \\ - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} - \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0),$$

for $r = 1, 2, \dots$.

Lemma 2.

$$(2) \quad \int_{-1}^1 v^i \rho^{(r)}(v) dv = \begin{cases} 0, & 0 \leq i < r, \\ (-1)^r r!, & i = r \end{cases}$$

for $r = 0, 1, 2, \dots$.

The following theorem was proved in [5].

Theorem 1. *The neutrix product $x^{-r} \square x^{-s}$ exists and*

$$x^{-r} \square x^{-s} = x^{-r-s}$$

for $r, s = 1, 2, \dots$.

The limits involved in the proof of Theorem 1 were easily evaluated. However, in the following, we are going to consider the neutrix product

$x_+^{-r} \square x_+^{-s}$. For this neutrix product, the limits are more complicated and so we only prove the existence of the limits and thus the existence of the neutrix product $x_+^{-r} \square x_+^{-s}$.

We now prove the following theorem.

Theorem 2. *The neutrix product $x_+^{-r} \square x_+^{-s}$ exists for $r, s = 1, 2, \dots$.*

PROOF. We first of all consider the case $s = 1$ and put

$$(x_+^{-r})_n = x_+^{-r} * \delta_n(x) = \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n^{(r)}(t) dt,$$

for $r = 1, 2, \dots$. Then

$$\begin{aligned} (3) \quad & (-1)^{r-1} (r-1)! \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^k dx \\ &= \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_t^{1/n} \delta'_n(s) \int_s^{1/n} x^k \ln(x-t) \ln(x-s) dx ds dt \\ &\quad + \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^t \delta'_n(s) \int_t^{1/n} x^k \ln(x-t) \ln(x-s) dx ds dt \\ &\quad + \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^{1/n} \delta'_n(s) \int_{1/n}^1 x^k \ln(x-t) \ln(x-s) dx ds dt \\ &= n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_v^1 \rho'(u) \\ &\quad \times \int_u^1 w^k \ln[(w-v)/n] \ln[(w-u)/n] dw du dv \\ &\quad + n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^v \rho'(u) \\ &\quad \times \int_v^1 w^k \ln[(w-v)/n] \ln[(w-u)/n] dw du dv \\ &\quad + n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) \\ &\quad \times \int_1^n w^k \ln[(w-v)/n] \ln[(w-u)/n] dw du dv = I_1 + I_2 + I_3, \end{aligned}$$

where the substitutions $ns = u$, $nt = v$ and $nx = w$ have been made.

It follows immediately that

$$(4) \quad \text{N-lim}_{n \rightarrow \infty} I_1 = \text{N-lim}_{n \rightarrow \infty} I_2 = 0,$$

for $k = 0, 1, 2, \dots, r-1$.

Now

$$(5) \quad \begin{aligned} & \int_1^n w^k \ln[(w-v)/n] \ln[(w-u)/n] dw \\ &= \int_1^n w^k [\ln(w-v) - \ln n] [\ln(w-u) - \ln n] dw \\ &= \ln^2 n \int_1^n w^k dw - 2 \ln n \int_1^n w^k \ln(w-v) dw \\ & \quad + \int_1^n w^k \ln(w-v) \ln(w-u) dw \end{aligned}$$

and it follows immediately that

$$(6) \quad \text{N-lim}_{n \rightarrow \infty} n^{r-k} \ln^2 n \int_1^n w^k dw = 0$$

for $k = 0, 1, 2, \dots$.

Further, by expanding $\ln(w-v)$ in powers of v/w , it also follows that

$$(7) \quad \text{N-lim}_{n \rightarrow \infty} n^{r-k} \ln n \int_1^n w^k \ln(w-v) dw = 0$$

for $k = 0, 1, 2, \dots$.

Finally, we have

$$(8) \quad \begin{aligned} & \int_1^n w^k \ln(w-v) \ln(w-u) dw \\ &= \int_1^n w^k \left[\ln w - \sum_{i=1}^{\infty} \frac{v^i}{i w^i} \right] \left[\ln w - \sum_{j=1}^{\infty} \frac{u^j}{j w^j} \right] dw \\ &= \int_1^n w^k \ln^2 w dw - 2 \sum_{i=1}^{\infty} \frac{v^i}{i} \int_1^n w^{k-i} \ln w dw \\ & \quad + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{v^i u^j}{i j} \int_1^n w^{k-i-j} dw \end{aligned}$$

and it follows that

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} n^{r-k} \int_1^n w^k \ln(w-v) \ln(w-u) dw \\ = - \sum_{j=1}^r \frac{v^{r-j+1} u^j}{j(r-k)(r-j+1)}. \end{aligned}$$

Hence

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) \int_1^n w^k \ln(w-v) \ln(w-u) dw du dv \\ (9) \quad = \frac{(-1)^r (r-1)!}{r-k}, \end{aligned}$$

for $k = 0, 1, 2, \dots, r-1$ on using equation (2). It follows from equations (5), (6), (7) and (9) that

$$(10) \quad \text{N-lim}_{n \rightarrow \infty} I_3 = \frac{(-1)^r (r-1)!}{r-k}.$$

It now follows from equations (3), (4) and (10) that

$$(11) \quad \text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^k dx = -\frac{1}{r-k},$$

for $k = 0, 1, 2, \dots, r-1$.

We now deal with the case $k = r$. Equation (3) still holds but this time it follows that

$$\begin{aligned} (12) \quad \text{N-lim}_{n \rightarrow \infty} I_1 = \int_{-1}^1 \rho^{(r)}(v) \int_v^1 \rho'(u) \int_u^1 w^r \ln |(w-v)| \\ \times \ln |(w-u)| dw du dv, \end{aligned}$$

$$\begin{aligned} (13) \quad \text{N-lim}_{n \rightarrow \infty} I_2 = \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^v \rho'(u) \int_v^1 w^r \ln |(w-v)| \\ \times \ln |(w-u)| dw du dv. \end{aligned}$$

Further, equation (8) is replaced by the equation

$$\begin{aligned} \int_1^n w^r \ln(w-v) \ln(w-u) dw &= \int_1^n w^r \ln^2 w dw \\ -2 \sum_{i=1}^{\infty} \frac{v^i}{i} \int_1^n w^{r-i} \ln w dw &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{v^i u^j}{ij} \int_1^n w^{r-i-j} dw. \end{aligned}$$

It follows that

$$\text{N-lim}_{n \rightarrow \infty} \int_1^n w^r \ln(w-v) \ln(w-u) dw = g_r(u, v),$$

say and so

$$(14) \quad \text{N-lim}_{n \rightarrow \infty} I_3 = \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) g_r(u, v) du dv.$$

We therefore see from equations (3), (12), (13) and (14) that

$$\text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^r dx$$

exists and we put

$$(15) \quad \text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^r dx = L_{r,1}.$$

When $k = r + 1$, it follows as for equation (3) that for any continuous function ψ

$$\begin{aligned} &\int_{-1/n}^{1/n} (x_+^{-r})_n (x_+^{-1})_n x^{r+1} \psi(x) dx \\ &= n^{-1} \int_{-1}^1 \rho^{(r)}(v) \int_v^1 \rho'(u) \int_u^1 w^{r+1} \psi(w/n) \ln |(w-v)/n| \\ &\quad \times \ln |(w-u)/n| dw du dv \\ &+ n^{-1} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^v \rho'(u) \int_u^1 w^{r+1} \psi(w/n) \ln |(w-v)/n| \\ &\quad \times \ln |(w-u)/n| dw du dv \end{aligned}$$

and it follows that

$$(16) \quad \lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} (x_+^{-r})_n (x_+^{-1})_n x^{r+1} \psi(x) dx = 0.$$

Next, when $x \geq 1/n$, we have

$$\begin{aligned} (-1)^{r-1} (r-1)! (x_+^{-r})_n &= \int_{-1/n}^{1/n} \ln |x-t| \delta_n^{(r)}(t) dt \\ &= n^r \int_{-1}^1 \ln |x-v/n| \rho^{(r)}(v) dv \\ &= n^r \int_{-1}^1 \left[\ln x - \sum_{i=1}^{\infty} \frac{v^i}{i n^i x^i} \right] \rho^{(r)}(v) dv \\ &= - \sum_{i=r}^{\infty} \int_{-1}^1 \frac{v^i}{i n^{i-r} x^i} \rho^{(r)}(v) dv. \end{aligned}$$

It follows that

$$|(r-1)! (x_+^{-r})_n| \leq \sum_{i=r}^{\infty} \int_{-1}^1 \frac{|v|^i}{i n^{i-r} x^i} |\rho^{(r)}(v)| dv \leq \sum_{i=r}^{\infty} \frac{K_r}{i n^{i-r} x^i},$$

where

$$K_r = \int_{-1}^1 |\rho^{(r)}(v)| dv$$

for $r = 1, 2, \dots$

If now $n^{-1} < \eta < 1$, then

$$\begin{aligned} (r-1)! \int_{1/n}^{\eta} |(x_+^{-r})_n (x_+^{-1})_n x^{r+1}| dx \\ \leq K_1 K_r \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \int_{1/n}^{\eta} \frac{n^{r-i-j+1} x^{r-i-j-1}}{ij} dx \\ = \frac{K_1 K_r}{n} \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \int_1^{n\eta} \frac{w^{r-i-j+1}}{ij} dw \end{aligned}$$

$$= \frac{K_1 K_r}{n} \left[\frac{\ln w}{r+1} + \frac{\ln w}{2r} + \sum_{i=r}^{\infty} \sum_{\substack{j=1 \\ i+j \neq r+2}}^{\infty} \frac{w^{r-i-j+2}}{ij(r-i-j+2)} \right]_1^{n\eta}$$

and it follows that

$$\lim_{n \rightarrow \infty} \int_{1/n}^{\eta} |(x_+^{-r})_n (x_+^{-1})_n x^{m+r}| dx \leq \frac{K_1 K_r \eta}{r!}$$

for $r = 1, 2, \dots$

Thus, if ψ is a continuous function

$$(17) \quad \lim_{n \rightarrow \infty} \left| \int_{1/n}^{\eta} (x_+^{-r})_n (x_+^{-1})_n x^{r+1} \psi(x) dx \right| = O(\eta)$$

for $r = 1, 2, \dots$

Now let φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^r \frac{x^k \varphi^{(k)}(0)}{k!} + \frac{x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!},$$

where $0 < \xi < 1$. Thus

$$\begin{aligned} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle &= \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n \varphi(x) dx \\ &= \sum_{k=0}^r \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^k dx \\ &\quad + \int_{-1/n}^{1/n} \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!} dx \\ &\quad + \int_{1/n}^{\eta} \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!} dx \\ &\quad + \int_{\eta}^1 \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!} dx. \end{aligned}$$

On using the equations (11), (15), (16) and (17) and noting that the sequence $\{(x_+^{-r})_n (x_+^{-1})_n\}$ converges uniformly to x^{-r-1} on the interval

$[\eta, 1]$, it follows that

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle \\ &= - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} + \int_{\eta}^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} dx + O(\eta), \end{aligned}$$

but since η can be made arbitrarily small, it follows that

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle \\ &= - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} + \int_0^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} dx \\ &= \int_0^1 x^{-r-1} \left[\varphi(x) - \sum_{k=0}^r \frac{x^k \varphi^{(k)}(0)}{k!} \right] dx - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} \\ &= \langle x_+^{-r-1}, \varphi(x) \rangle + \frac{(-1)^r}{r!} [L_{r,1} + \phi(r)] \langle \delta^{(r)}(x), \varphi(x) \rangle \end{aligned}$$

on using equation (1).

The neutrix product $x_+^{-r} \square x_+^{-1}$ therefore exists and

$$x_+^{-r} \square x_+^{-1} = x_+^{-r-1} + \frac{(-1)^r}{r!} [L_{r,1} + \phi(r)] \delta^{(r)}(x)$$

on the interval $[-1, 1]$. However, the product $x_+^{-r} \cdot x_+^{-1}$ obviously exists on any interval not containing the origin, and so the neutrix product $x_+^{-r} \square x_+^{-1}$ exists on the real line for $r = 1, 2, \dots$

Suppose now that $x_+^{-r} \square x_+^{-s}$ exists and is of the form

$$x_+^{-r} \square x_+^{-s} = x_+^{-r-s} + a_{r,s} \delta^{(r+s-1)}(x)$$

for $r = 1, 2, \dots$ and for some positive integer s . Then the derivative of $x_+^{-r} \square x_+^{-s}$ exists, and

$$\begin{aligned} (x_+^{-r} \square x_+^{-s})' &= -(r+s) x_+^{-r-s-1} + a_{r,s} \delta^{(r+s)}(x) \\ &= -s x_+^{-r} \square x_+^{-s-1} - r x_+^{-r-1} \square x_+^{-s} \\ &= -s x_+^{-r} \square x_+^{-s-1} - r x_+^{-r-s-1} - r a_{r+1,s} \delta^{(r+s)}(x). \end{aligned}$$

The product $x_+^{-r} \square x_+^{-s-1}$ therefore exists and

$$\begin{aligned} x_+^{-r} \square x_+^{-s-1} &= x_+^{-r-s-1} - \frac{ra_{r+1,s} + a_{r,s}}{s} \delta^{(r+s)}(x) \\ &= x_+^{-r-s-1} + a_{r,s+1} \delta^{(r+s)}(x). \end{aligned}$$

It follows by induction that the product $x_+^{-r} \square x_+^{-s}$ exists for $r, s = 1, 2, \dots$

Defining the distribution x_-^{-r} by

$$x_-^{-r} = (-x)_+^{-r}$$

for $r = 1, 2, \dots$, we have □

Corollary 2.1. *The neutrix product $x_-^{-r} \square x_-^{-s}$ exists for $r, s = 1, 2, \dots$*

PROOF. With the above notation we have

$$(18) \quad x_+^{-r} \square x_+^{-s} = x_+^{-r-s} + a_{r,s} \delta^{(r+s-1)}(x).$$

Replacing x in this equation by $-x$, we get

$$(19) \quad x_-^{-r} \square x_-^{-s} = x_-^{-r-s} - (-1)^{r+s} a_{r,s} \delta^{(r+s-1)}(x),$$

proving the existence of neutrix product $x_-^{-r} \square x_-^{-s}$. □

Corollary 2.2.

$$(20) \quad x_+^{-r} \square x_+^{-s} + (-1)^{r+s} x_-^{-r} \square x_-^{-s} = x^{-r-s}$$

for $r, s = 1, 2, \dots$

PROOF. Equation (20) follows immediately from equations (18) and (19). □

Theorem 3. *The neutrix product $x_+^{-r} \square \ln x_+$ exists for $r = 1, 2, \dots$. In particular, the product $x_+^{-1} \cdot \ln x_+$ exists and*

$$(21) \quad x_+^{-1} \cdot \ln x_+ = x_+^{-1} \ln x_+.$$

PROOF. We put

$$(\ln x_+)_n = \ln x_+ * \delta_n(x) = \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n(t) dt$$

and

$$(x_+^{-1})_n = x_+^{-1} * \delta_n(x) = \int_{-1/n}^{1/n} \ln(x-t)_+ \delta'_n(t) dt.$$

Since $\ln x_+$ and $\ln^2 x_+$ are locally summable functions, it follows that

$$\lim_{n \rightarrow \infty} (\ln x_+)_n^2 = \ln^2 x_+.$$

Thus, for arbitrary φ in $\mathcal{D}$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle [(\ln x_+)_n^2]', \varphi(x) \rangle &= 2 \lim_{n \rightarrow \infty} \langle (\ln x_+)_n (x_+^{-1})_n, \varphi(x) \rangle \\ &= \langle (\ln x_+^2)', \varphi(x) \rangle = 2 \langle x_+^{-1} \ln x_+, \varphi(x) \rangle \end{aligned}$$

and equation (21) follows.

Now suppose that the neutrix product $x_+^{-r} \square \ln x_+$ exists and is of the form

$$x_+^{-r} \square \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0} \delta^{(r-1)}(x)$$

for some positive integer r . Then the derivative of $x_+^{-r} \square \ln x_+$ exists and

$$\begin{aligned} (x_+^{-r} \square \ln x_+)' &= -r x_+^{-r-1} \ln x_+ + x_+^{-r-1} + a_{r,0} \delta^{(r)}(x) \\ &= -r x_+^{-r-1} \square \ln x_+ + x_+^{-r} \square x_+^{-1} \\ &= -r x_+^{-r-1} \square \ln x_+ + x_+^{-r-1} + a_{r,1} \delta^{(r)}(x). \end{aligned}$$

The product $x_+^{-r-1} \square \ln x_+$ therefore exists and

$$\begin{aligned} x_+^{-r-1} \square \ln x_+ &= x_+^{-r-1} \ln x_+ + \frac{a_{r,1} - a_{r,0}}{r} \delta^{(r)}(x) \\ &= x_+^{-r-1} \ln x_+ + a_{r+1,1} \delta^{(r)}(x). \end{aligned}$$

It follows by induction that the product $x_+^{-r} \square \ln x_+$ exists for $r = 1, 2, \dots$

□

Corollary 3.1. *The neutrix product $x_-^{-r} \square \ln x_-$ exists for $r = 1, 2, \dots$. In particular, the product $x_-^{-1} \cdot \ln x_-$ exists and*

$$(22) \quad x_-^{-1} \cdot \ln x_- = x_-^{-1} \ln x_-.$$

PROOF. With the above notation we have

$$(23) \quad x_+^{-r} \square \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0} \delta^{(r-1)}(x).$$

Replacing x in this equation by $-x$, we get

$$(24) \quad x_-^{-r} \square \ln x_- = x_-^{-r} \ln x_- - (-1)^r a_{r,0} \delta^{(r-1)}(x),$$

proving the existence of neutrix product $x_-^{-r} \square \ln x_-$. The particular case $r = 1$ of course reduces to the product

$$x_-^{-1} \cdot \ln x_- = x_-^{-1} \ln x_- \quad \square$$

Corollary 3.2.

$$(25) \quad x_+^{-r} \square \ln x_+ + (-1)^r x_-^{-r} \square \ln x_- = x^{-r} \ln |x|$$

for $r = 1, 2, \dots$

PROOF. Equation (25) follows immediately from equations (23) and (24). $\square$

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