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On the characterization of additive functions on Gaussian integers

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Abstract. Let f denote an additive function on the Gaussian integers. We prove some theorems of characterization with linear and quadratic arguments. If e.g. for a completely additive function $f(a\alpha+b)-tf(\alpha)=c$ or $f(\alpha^2+1)+f(\alpha^2-1)=c$, then $f(\alpha)=0$ for all nonzero Gaussian integers α.

In 1946 Erdős [2] proved the following theorems:

Theorem 1 (Erdős). *Let f be a real valued additive function. If $f(n+1)-f(n)\to 0$, then $f(n)=c\log n$ for all $n\in\mathbb{N}$.*

Theorem 2 (Erdős). *If a real valued additive function f is monotonically increasing, then $f(n)=c\log n$ for all $n\in\mathbb{N}$.*

I. Kátai [3] generalized Theorem 1 for completely additive functions using a result of E. Wirsing [6]:

Theorem 3 (Kátai). *Let f be a completely additive function. If $\sum_{i=1}^{m} c_i f(n+a_i)=o(\log n)$, then $f(n)=c\log n$ for all $n\in\mathbb{N}$ or $f=0$.*

The following generalizations are due to P.D.T.A. Elliott [1] and myself ([4], [5]):

Theorem 4 (Elliott, [1]). *Let f be an additive function, $A>0$, $C>0$, B, D integers and $\Delta_1=AC(AD-BC)\neq 0$. If $f(An+B)-f(Cn+D)\to c$, then $f(n)=c'\log n$ for all (n,Δ_1).*

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Theorem 5 [5]**.** *Let f be a completely additive function. If $f(2n+A)-f(n)$ is monotonic from some number on, then $f(n)=c\log n$ for all $n\in\mathbb{N}$.*

Theorem 6 [4]**.** *Let f denote a completely additive function. If $f(n^2+1)=s_1f(n)+s_2f(n-1)+o(\log n)$ (s_1, s_2 are not both zero), then $f(n)=c\log n$.*

In this article we intend to prove some similar results on the set of the Gaussian integers. Let G^* denote the set of the nonzero Gaussian integers. Let α, β be the elements of this set and $N(\alpha):=\alpha\overline{\alpha}$.

Definition 1. The function f is G-additive, if $f(\alpha\beta)=f(\alpha)+f(\beta)$ for all relatively prime $\alpha,\beta\in G^*$.

Definition 2. The function f is completely G-additive, if $f(\alpha\beta)=f(\alpha)+f(\beta)$ for all $\alpha,\beta\in G^*$.

Remark. We can prove easily that $f(\epsilon)=0$ for any additive f and arbitrary Gaussian unit ϵ.

We prove the following results:

Theorem 7. *Let a, b denote some fixed elements of G^* and let $t\in\mathbb{C}\setminus\{0\}$.*

(i) *If for a G-additive function*

$$f(a\alpha+b)-tf(\alpha)\to c, \tag{1}$$

then $f(n)=c'\log n$ for all $n\in\mathbb{N}^+$ coprime to $2N(ab)$.

(ii) *If for a completely G-additive function*

$$f(a\alpha+b)-tf(\alpha)=c, \tag{1'}$$

then $f(\alpha)=0$ for all $\alpha\in G^*$.

Theorem 8. *If for a completely G-additive function*

$$f(\alpha^2+1)+f(\alpha^2-1)=c,$$

then $f(\alpha)=0$ for all $\alpha\in G^*$.

Theorem 9. (i) *If for a completely G-additive function*

$$f(\alpha^2+1) = s_1 f(\alpha) + s_2 f(\alpha-1) + o(\log N(\alpha)) \quad (s_1, s_2 \text{ are not both zero}),$$

then $f(z) = 0$ *for all* $z \in \mathbb{Z} \setminus \{0\}$.

(ii) *If for a completely G-additive function*

$$f(\alpha^2+1) = s_1 f(\alpha) + s_2 f(\alpha-1) + c \quad (s_1, s_2 \text{ are not both zero}),$$

then $f(\alpha) = 0$ *for all* $\alpha \in G^*$.

Proofs

Proof of Theorem 7. (i) If f is G-additive, then by substituting $\overline{a}bN(b)\alpha$ into (1) we have

$$f(N(ab)\alpha+1) - tf(\alpha) \to C', \tag{2}$$

for all α coprime to $N(ab)$ with $C' = c + tf(\overline{a}bN(b)) - f(b)$. By substituting 2α into (2) we have

$$f(2N(ab)\alpha+1) - tf(\alpha) \to C'' \tag{3}$$

for all α coprime to $2N(ab)$ with $C'' = c + tf(2\overline{a}bN(b)) - f(b)$. The difference of (3) and (2) shows that

$$f(2N(ab)\alpha+1) - f(N(ab)\alpha+1) \to C''' \tag{4}$$

for all α coprime to $2N(ab)$. By substituting $2N(ab)\alpha+1$ into (4) we have that

$$f(4N^2(ab)\alpha + 2N(ab)+1) - f(2N^2(ab)\alpha + N(ab)+1) \to C'''$$

for all $\alpha \in G^*$. Applying Theorem 4 we get that $f(n) = c' \log n$ for all $n \in \mathbb{N}$ coprime to $2N(ab)$.

(ii) Let f be a completely G-additive function.

By substituting $b\alpha$ into (1') we have

$$f(a\alpha+1) = tf(\alpha) + c_1 \tag{5}$$

with $c_1 = c + (t-1)f(b)$. Therefore

$$f((a\alpha+1)^2) = f(a[\alpha(a\alpha+2)]+1) = tf(\alpha) + tf(a\alpha+2) + c_1$$

and

$$f((a\alpha+1)^2) = 2f(a\alpha+1) = 2tf(\alpha) + 2c_1,$$

i.e.

$$f(a\alpha+2) = f(\alpha) + c_1/t. \tag{6}$$

By substituting 2α into (6) we get

$$f(a\alpha+1) = f(\alpha) + c_1/t. \tag{7}$$

By the comparison of (5) and (7) f is constant, i.e. $f(\alpha) = 0$ for all $\alpha \in G^*$ or $t = 1$. If $t = 1$, then we also prove , that $f(\alpha) = 0$ for all $\alpha \in G^*$. First we prove by induction, that

$$f(a\alpha+s) = f(\alpha) + c_1 \tag{8}$$

for all $s \in \mathbb{N}^+$. For $s = 1$ it is true by (7). By the assumption of induction for $s \leq z$ we have

$$f((a\alpha+1)(a\alpha+z)) = f(a\alpha+1) + f(a\alpha+z) = 2f(\alpha) + 2c_1$$

and

$$f(a[\alpha(a\alpha+z+1)]+z) = f(\alpha) + f(a\alpha+z+1) + c_1,$$

which follow $f(a\alpha+z+1) = f(\alpha) + c$. By substituting $s = a$ into (8) we have $f(\alpha+1) = f(\alpha) + c_1 - f(a)$. By restricting α to the natural numbers $f(n) - f(n-1) = c_1 - f(a)$ for all $n \in \mathbb{N}$. By substituting n^2 here we have $c_1 - f(a) = f(n^2) - f(n^2-1) = [f(n) - f(n-1)] - [f(n+1) - f(n)] = 0$, i.e. $c_1 = f(a)$. Applying Theorem 1 $f(n+1) = f(n)$ follows $f(n) = 0$ for all $n \in \mathbb{N}$. Using that $f(\alpha+1) = f(\alpha)$ for all $\alpha \in G^*$, we prove by induction that $f(\delta) = 0$ also for all not real Gaussian primes δ. By the Remark it is enough to consider the Gaussian primes of form $\pi = 1 + i$ and $\pi = x + yi$ with even number x and odd number y as $f(\pi) + f(\overline{\pi}) = f(N(\pi)) = 0$ and $f(i\pi) = f(-i\pi) = f(-\pi) = f(\pi)$. We have $0 = f(2) = 2f(1+i)$. For any other π the Gaussian integer $\pi - 1$ is divisible by $1 + i$, i.e. it is not a Gaussian prime. We assume that $f(\gamma) = 0$ for all Gaussian-primes which norm is less then $f(\pi)$. As $f(\pi)=f(\pi-1)$ and $f(\beta)=0$ for all prime divisors β of $\pi-1$ by the hypotesis of the induction, therefore $f(\pi)=0$ is also satisfied as f is a completely G-additive function. $\square$

Proof of Theorem 8. As $\alpha^2+1=(\alpha+i)(\alpha-i)$, we have

$$f(\alpha+i)+f(\alpha-i)+f(\alpha+1)+f(\alpha-1)=c. \tag{9}$$

By substituting $\alpha-i$ into (9), $(1+i)\alpha$ into (10), $2\alpha+i$ and $(1-i)\alpha+i$ into (11) we have that

$$f(\alpha)+f(\alpha-2i)+f(\alpha+1-i)+f(\alpha-1-i)=c, \tag{10}$$

$$f(\alpha)+f(\alpha-1-i)+f(\alpha-i)+f(\alpha-1)=c_1, \tag{11}$$

$$f(2\alpha+i)+f(2\alpha-1)+f(\alpha)+f(2\alpha-1+i)=c_2 \tag{12}$$

and

$$f(2\alpha-1+i)+f(2\alpha-1-i)+f(\alpha)+f(\alpha-1)=c_3. \tag{13}$$

The difference of (12) and (13) shows that

$$f(2\alpha+i)-f(2\alpha-1-i)+f(2\alpha-1)-f(\alpha-1)=c_4. \tag{14}$$

By substituting 2α into (11) and (9) we have

$$f(\alpha)+f(2\alpha-1-i)+f(2\alpha-i)+f(2\alpha-1)=c_1 \tag{15}$$

and

$$f(2\alpha+i)+f(2\alpha-i)+f(2\alpha+1)+f(2\alpha-1)=c. \tag{16}$$

The difference of (15) and (16) shows that

$$f(2\alpha+i)-f(2\alpha-1-i)+f(2\alpha+1)-f(\alpha)=c_5. \tag{17}$$

By the comparison of (14) and (17) we get

$$f(2\alpha+1)-f(\alpha)=f(2\alpha-1)-f(\alpha-1)+c_6. \tag{18}$$

By restricting α to natural numbers (18) follows that $f(2n+1)-f(n)$ is monotonic. Applying Theorem 5 we obtain $f(n)=0$ for all $n\in\mathbb{N}$ and also that $c_6=0$. We prove by induction that $f(\pi)=0$ also for all not real Gaussian primes. As the norm of $2\alpha+1$ is greater than the norm of any other argument in (18) and the minimal value of $N(2\alpha+1)$ is 13, it is enough to verify that $f(1+i) = f(2+i) = f(2+3i) = 0$ using the Remark. It is true as $0 = f(2) = 2f(1+i)$ and $\alpha = 1+i$ and $\alpha = i$ in (18) imply $f(1+2i) = f(3+2i)$ and $f(1+2i) = f(1+i)$. $\square$

Proof of Theorem 9. (i) is a direct consequence of Theorem 6 and the Remark.

(ii) By induction we prove that $f(x+yi) = 0$ for all $x \in \mathbb{Z}$ and $y \in \mathbb{N}$. For $y = 0$ it is true by (i) (x can be arbitrarily choosen). Let us assume that it is true for all $0 \le y \le s$. If we substitute $x + si$ in the condition of the theorem we get

$$f(x+(s+1)i) + f(x+(s-1)i) = sf(x+si) + tf(x-1+si),$$

i.e. by the assumption of the induction $f(x+(s+1)i) = 0$. If $y < 0$, then $f(x+iy) = f(-x-iy) = 0$ as $-y \in \mathbb{N}$. $\square$

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