

On a certain application of Patterson's curvature identity

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Abstract. We present curvature properties of four-dimensional semi-Riemannian manifolds satisfying some condition of pseudosymmetry type. We prove that every such manifold with non-zero associated function L is pseudosymmetric, its scalar curvature does not vanish and L must be equal to $\frac{1}{3}$. We also describe non-trivial example of a manifold realizing all these conditions.

1. Introduction

Let (M, g) be a connected n -dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞ . We denote by ∇ , R , C , S and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M, g) , respectively.

E. M. PATTERSON [13] has given (among others algebraic identities satisfied by the curvature tensor) the following result

Proposition 1.1. *The Weyl conformal curvature tensor C of every 4-dimensional Riemannian manifold satisfies the identity*

$$g_{hm}C_{lijk} + g_{lm}C_{ihjk} + g_{im}C_{hljk} + g_{hj}C_{likm} + g_{lj}C_{ihkm} \\
+ g_{ij}C_{hlkm} + g_{hk}C_{limj} + g_{lk}C_{ihmj} + g_{ik}C_{hlmj} = 0.$$

The above identity plays an important role in investigations of the curvature properties of 4-dimensional semi-Riemannian manifolds. Among

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others, the authors of the present paper have used the Patterson's identity during the study of 4-dimensional manifolds of pseudosymmetry type ([3], [8]).

A semi-Riemannian manifold (M, g) is said to be semisymmetric [14] if $R \cdot R = 0$ holds on M . As a proper generalization of locally symmetric spaces ($\nabla R = 0$) semisymmetric manifolds were studied by many authors. In the Riemannian case, Z. I. SZABÓ obtained in the early eighties a full intrinsic classification of semisymmetric Riemannian manifolds [14]. Very recently a theory of Riemannian semisymmetric manifolds has been presented in the monograph [1].

The profound investigation of several properties of semisymmetric manifolds, gave rise to their next generalization: the pseudosymmetric manifolds.

A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric* ([6], Section 3.1) if at every point of M the following condition is satisfied:

$(*)_1$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

This condition is equivalent to the relation $R \cdot R = L_R Q(g, R)$ on the set $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on $\mathcal{U}_R$. The definitions of the tensors used will be given in Section 2. There exist various examples of pseudosymmetric manifolds which are non-semisymmetric and a review of results on pseudosymmetric manifolds is given in [6] (see also [15]).

It is easy to see that if $(*)_1$ holds on a semi-Riemannian manifold (M, g) , then at every point of M the following condition is satisfied:

$(*)_2$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The converse statement is not true ([6], Section 8.1). A semi-Riemannian manifold (M, g) is called *Ricci-pseudosymmetric* ([6], Section 4.1) if at every point of M the condition $(*)_2$ is fulfilled. If a manifold (M, g) is Ricci-pseudosymmetric then the relation $R \cdot S = L_S Q(g, S)$ holds on the set $\mathcal{U}_S = \{x \in M \mid S \neq \frac{\kappa}{n} g \text{ at } x\}$, where L_S is a certain function on $\mathcal{U}_S$.

It is easy to verify that if $(*)_1$ holds on a semi-Riemannian manifold (M, g) , $n \geq 4$, then at every point of M the following condition is satisfied:

$(*)_3$ the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

The converse statement is not true (cf. Example 4.1; see also [6], Section 9.3).

A semi-Riemannian manifold (M, g) , $n \geq 4$, is called *Weyl-pseudo-symmetric* if at every point of M the condition $(*)_3$ is fulfilled. If a manifold (M, g) is Weyl-pseudosymmetric then the relation $R \cdot C = L_C Q(g, C)$ holds on the set $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on $\mathcal{U}_C$.

Evidently, every semi-Riemannian semisymmetric manifold realizes trivially at every point the following condition ([9])

(*) the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent.

This condition is equivalent to the relation

$$(1) \quad R \cdot C = L Q(S, C)$$

on the set $\mathcal{U} = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$, for a certain function L on $\mathcal{U}$. There exist non-semisymmetric manifolds realizing (*) ([9], [10]).

Semi-Riemannian manifolds realizing $(*)_1$, $(*)_2$, $(*)_3$ and (*) or other conditions of this kind, are called *manifolds of pseudosymmetry type*.

Recently 4-dimensional warped product manifolds $M \times_F N$, $\dim M=1$, satisfying the condition (*) have been considered in [10]. In the present paper we investigate arbitrary 4-dimensional semi-Riemannian manifolds fulfilling (*). In Section 2 we fix the notations and present auxiliary lemmas. In Section 3 we consider 4-dimensional manifolds satisfying the equality $Q(S, C) = 0$ and we prove that such manifolds are semisymmetric and their scalar curvature is equal to zero. In Section 4 we investigate 4-dimensional manifolds satisfying (1) with $L \neq 0$. We prove, among others, that every such manifold is pseudosymmetric with $L_R = \frac{\kappa}{12}$, its scalar curvature does not vanish and the associated function L must be equal to $\frac{1}{3}$. Finally, we describe an example of a manifold having all these properties.

2. Preliminaries

Let (M, g) be an n -dimensional, $n \geq 3$, semi-Riemannian manifold. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M . Next, we define

the endomorphisms $\mathcal{R}(X, Y)$, $\mathcal{C}(X, Y)$ and $X \wedge Y$ of $\Xi(M)$ by

$$\begin{aligned}\mathcal{R}(X, Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \\ \mathcal{C}(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{n-2} \left(X \wedge \mathcal{S}Y + \mathcal{S}X \wedge Y - \frac{\kappa}{n-1} X \wedge Y \right) Z, \\ (X \wedge Y)Z &= g(Y, Z)X - g(X, Z)Y,\end{aligned}$$

respectively, where $X, Y, Z \in \Xi(M)$. Now the Riemann–Christoffel curvature tensor R , the Weyl conformal curvature tensor C and the (0,4)-tensor G of (M, g) are defined by

$$\begin{aligned}R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4), \\ C(X_1, X_2, X_3, X_4) &= g(\mathcal{C}(X_1, X_2)X_3, X_4), \\ G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge X_2)X_3, X_4),\end{aligned}$$

respectively. A tensor $\mathcal{B}$ of type $(1, 3)$ on M is said to be a generalized curvature tensor if

$$\begin{aligned}\bigoplus_{X_1, X_2, X_3} \mathcal{B}(X_1, X_2)X_3 &= 0, \quad \mathcal{B}(X_1, X_2) + \mathcal{B}(X_2, X_1) = 0, \\ B(X_1, X_2, X_3, X_4) &= B(X_3, X_4, X_1, X_2),\end{aligned}$$

where $B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4)$.

For an $(0, 2)$ -tensor field A on (M, g) we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y$, where $X, Y, Z \in \Xi(M)$. In particular we have $X \wedge_g Y = X \wedge Y$.

For an $(0, k)$ -tensor field T , $k \geq 1$, an $(0, 2)$ -tensor field A and a generalized curvature tensor $\mathcal{B}$ on (M, g) we define the tensors $B \cdot T$ and $Q(A, T)$ by

$$\begin{aligned}(B \cdot T)(X_1, \dots, X_k; X, Y) &= -T(\mathcal{B}(X, Y)X_1, X_2, \dots, X_k) \\ &\quad - \dots - T(X_1, \dots, X_{k-1}, \mathcal{B}(X, Y)X_k), \\ Q(A, T)(X_1, \dots, X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) \\ &\quad - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k),\end{aligned}$$

where $X, Y, Z, X_1, X_2, \dots \in \Xi(M)$. Putting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we obtain the tensors

$R \cdot R, R \cdot C, R \cdot S, C \cdot S, Q(g, R), Q(g, C), Q(g, S)$ and $Q(S, C)$, respectively. Let (M, g) be a semi-Riemannian manifold covered by a system of charts $\{W; x^k\}$.

We denote by $g_{ij}, R_{hijk}, S_{ij}, G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$(2) \quad C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\ + \frac{\kappa}{(n-1)(n-2)}G_{hijk}$$

the local components of the metric tensor g , the Riemann-Christoffel curvature tensor R , the Ricci tensor S , the tensor G and the Weyl tensor C , respectively. Further, we denote by $S_{ij}^2 = S_{ir}S_j^r$ and $S_i^j = g^{jr}S_{ir}$ the local components of the tensor S^2 defined by $S^2(X, Y) = S(SX, Y)$, and of the Ricci operator $\mathcal{S}$, respectively.

At the end of this section we present some results which will be used in the next sections.

Lemma 2.1 ([5], Lemma 1). *Let a tensor $A_{lmhs_1 \dots s_N}$ of type $(0, N+3)$ be symmetric in (l, m) and skew-symmetric in (m, h) . Then $A_{lmhs_1 \dots s_N} = 0$.*

Lemma 2.2 ([11], Lemma 2). *Let A and D be two symmetric $(0, 2)$ -tensors at a point x of a semi-Riemannian manifold (M, g) . If the condition $Q(A, D) = 0$ is fulfilled at x , then the tensors A and D are linearly dependent at x .*

Lemma 2.3 ([7], Theorem 1). *Let $\mathcal{B}$ be a generalized curvature tensor at $x \in M$ such that the condition*

$$\mathfrak{S}_{X,Y,Z} \omega(X)\mathcal{B}(Y, Z) = 0$$

is satisfied for $\mathcal{B}$ and a covector ω at x , where $X, Y, Z \in T_x(M)$ and $\mathfrak{S}$ denotes the cyclic sum. If $\omega \neq 0$ then the following relation holds at x : $B \cdot B = Q(\text{Ric}(\mathcal{B}), B)$.

Lemma 2.4 ([2], Proposition 4.1). *Let (M, g) , $\dim M \geq 3$, be a semi-Riemannian manifold. Let A be a non-zero symmetric $(0, 2)$ -tensor and $\mathcal{B}$ a generalized curvature tensor at a point x of M satisfying the condition $Q(A, B) = 0$.*

Moreover, let V be a vector at x such that the scalar $\rho = a(V)$ is non-zero, where a is the covector defined by $a(X) = A(X, V)$, $X \in T_x(M)$.

(i) If the tensor $A - \frac{1}{\rho}a \otimes a$ vanishes then the relation

$$\mathfrak{S}_{X,Y,Z} a(X)\mathcal{B}(Y, Z) = 0$$

holds at x , where $X, Y, Z \in T_x(M)$.

(ii) If the tensor $A - \frac{1}{\rho}a \otimes a$ is non-zero then the relation

$$\rho B(X, Y, Z, W) = \lambda(A(X, W)A(Y, Z) - A(X, Z)A(Y, W))$$

holds at x , where $\lambda \in \mathbb{R}$ and $X, Y, Z, W \in T_x(M)$.

Moreover, in both cases the following condition $B \cdot B = Q(\text{Ric}(\mathcal{B}), B)$ holds at x .

Lemma 2.5 ([12], Theorems 1 and 2). *Let (M, g) be a Weyl-pseudo-symmetric semi-Riemannian manifold satisfying the following condition*

$$\mathfrak{S}_{X,Y,Z} a(X)\mathcal{C}(Y, Z) = 0,$$

where a is a 1-form on M .

If $a \neq 0$ and $C \neq 0$ at a point $x \in M$, then the following relations are satisfied at x :

$$\begin{aligned} L_C &= \frac{\kappa}{n(n-1)}, & S(W, \mathcal{C}(X, Y)Z) &= \frac{\kappa}{n} C(X, Y, Z, W), \\ Q\left(S - \frac{\kappa}{n}g, C\right) &= 0, & R \cdot R &= L_C Q(g, R). \end{aligned}$$

3. Manifolds with vanishing tensor field $Q(S, C)$

Theorem 3.1. *Let (M, g) , $\dim M = 4$, be a semi-Riemannian manifold satisfying at a point x of M the equality $Q(S, C) = 0$. If $S \neq 0$ and $C \neq 0$ at x , then the following relations*

$$\kappa = 0, \quad R \cdot R = 0$$

hold at x . Moreover, there exists a non-zero covector a at x such that

$$\mathfrak{S}_{X,Y,Z} a(X)\mathcal{C}(Y, Z) = 0.$$

PROOF. It is easy to verify that the following identity is satisfied on M

$$\begin{aligned}
 (3) \quad (C \cdot C)_{hijklm} &= (R \cdot C)_{hijklm} \\
 &+ \frac{1}{n-2} \left(\frac{\kappa}{n-1} Q(g, C)_{hijklm} - Q(S, C)_{hijklm} \right) \\
 &- \frac{1}{n-2} (g_{hl} S_{mr} C_{ijk}^r - g_{hm} S_{lr} C_{ijk}^r - g_{il} S_{mr} C_{hjk}^r \\
 &+ g_{im} S_{lr} C_{hjk}^r + g_{jl} S_{mr} C_{khi}^r - g_{jm} S_{lr} C_{khi}^r \\
 &- g_{kl} S_{mr} C_{jhi}^r + g_{km} S_{lr} C_{jhi}^r).
 \end{aligned}$$

According to Lemma 2.4, we may consider two cases (we will use the notations of the mentioned lemma):

(i) $S = \frac{1}{\rho} a \otimes a$. In this case we have

$$(4) \quad a_l C_{hijk} + a_h C_{iljk} + a_i C_{lhjk} = 0,$$

which implies $a_r C_{ijk}^r = 0$ and consequently $S_{ir} C_{hjk}^r = 0$. Thus the equation $C \cdot C = 0$, which follows from Lemma 2.3, and our assumption turn (3) into $R \cdot C = -\frac{\kappa}{(n-1)(n-2)} Q(g, C)$. Applying now Lemma 2.5 we obtain $\kappa = 0$ and $R \cdot R = 0$.

Now we consider the case:

(ii) $S - \frac{1}{\rho} a \otimes a \neq 0$. In this case we have

$$(5) \quad \rho C_{hijk} = \lambda (S_{hk} S_{ij} - S_{hj} S_{ik}).$$

Contracting (5) with g^{ij} we get $S_{hk}^2 = \kappa S_{hk}$. On the other hand transvecting (5) with S_p^h and using the last equality we obtain

$$(6) \quad S_p^r C_{rijk} = \kappa C_{pijk}.$$

Transvecting now the Patterson's identity with S^{hm} , in virtue of (6), we immediately have $\kappa C_{lijk} = 0$. Thus we have $\kappa = 0$, which turns (6) into $S_p^r C_{rijk} = 0$. Transvecting the Patterson's identity with S_p^h , in view of the last equality we get $S_{pm} C_{lijk} + S_{pj} C_{likm} + S_{pk} C_{limj} = 0$, which immediately implies (4). Semisymmetry of M follows in the same manner as in the case (i). This completes the proof. $\square$

4. Manifolds satisfying the condition (1)

First we observe that if $L = 0$ at x then the relation (1) reduces to $R \cdot C = 0$. This shows that (M, g) is a so called Weyl-semisymmetric manifold. Such a manifold need not be semisymmetric, as is shown by the example below.

Example 4.1. Let (M, g) be the 4-dimensional manifold defined in [4] (Lemme 1.1). As was shown in [4] (see Lemme 1.1 and Remarqué 1.5), (M, g) is a non-conformally flat and non-semisymmetric, Weyl-semisymmetric manifold, i.e. the tensors C and $R \cdot R$ are non-zero and the condition $R \cdot C = 0$ holds on M .

Thus we restrict our considerations in this section to the open subset $\mathcal{U}_L \subset \mathcal{U}$ on which the associated function L does not vanish.

Lemma 4.1. *Let (M, g) be a 4-dimensional semi-Riemannian manifold satisfying the condition (1). If $L \neq 0$ at x , then the following relations are fulfilled at x :*

$$(7) \quad S_m^r C_{rijk} + S_j^r C_{rikm} + S_k^r C_{rimj} = 0,$$

$$(8) \quad C \cdot S = 0,$$

$$(9) \quad S_p^r C_{rikj} = S_k^r C_{rjpi}.$$

PROOF. Applying to the Patterson's identity the operation $R \cdot$ and using (1), in view of $L \neq 0$, we get

$$(10) \quad \begin{aligned} & g_{hm} Q(S, C)_{lijkpt} + g_{lm} Q(S, C)_{ihjkpt} + g_{im} Q(S, C)_{hljkpt} \\ & + g_{hj} Q(S, C)_{likmpt} + g_{lj} Q(S, C)_{ihkmpt} + g_{ij} Q(S, C)_{hlkmpt} \\ & + g_{hk} Q(S, C)_{limjpt} + g_{lk} Q(S, C)_{ihmjpt} + g_{ik} Q(S, C)_{hlmjpt} = 0. \end{aligned}$$

Using the definition of the tensor $Q(S, C)$, by a standard calculation, we obtain

$$\begin{aligned} & Q(S, C)_{lijkpm} + Q(S, C)_{likmpj} + Q(S, C)_{limjpk} \\ & = 2(S_{pm} C_{likj} + S_{pj} C_{limk} + S_{pk} C_{lijm}) \\ & - (S_{ml} C_{pijk} + S_{im} C_{lpjk} + S_{jl} C_{pikm} + S_{ij} C_{lpkm} + S_{kl} C_{pimj} + S_{ik} C_{lpmj}), \end{aligned}$$

$$g^{rs}Q(S, C)_{rljkps} = -\kappa C_{pljk} + S_p{}^r C_{rljk} - S_l{}^r C_{rpjk} - S_j{}^r C_{rlpk} - S_k{}^r C_{rljp}.$$

Contracting (10) with g^{ht} and using the above relations, we have

$$\begin{aligned}
(11) \quad & 2(S_{pm}C_{likj} + S_{pj}C_{limk} + S_{pk}C_{lijm}) - (S_{lm} - \kappa g_{lm})C_{pijk} \\
& - (S_{lj} - \kappa g_{lj})C_{pikm} - (S_{lk} - \kappa g_{lk})C_{pimj} - (S_{im} - \kappa g_{im})C_{lpjk} \\
& - (S_{ij} - \kappa g_{ij})C_{lpkm} - (S_{ik} - \kappa g_{ik})C_{lpmj} \\
& + S_i{}^r(g_{lm}C_{rpjk} + g_{lj}C_{rpkm} + g_{lk}C_{rpmj}) \\
& + S_l{}^r(g_{im}C_{rpjk} + g_{ij}C_{rpmk} + g_{ik}C_{rpjm}) \\
& + g_{im}(S_p{}^r C_{rljk} - S_j{}^r C_{rlpk} - S_k{}^r C_{rljp}) \\
& + g_{ij}(S_p{}^r C_{rlkm} - S_k{}^r C_{rlpm} - S_m{}^r C_{rlkp}) \\
& + g_{ik}(S_p{}^r C_{rlmj} - S_m{}^r C_{rlpj} - S_j{}^r C_{rlmp}) \\
& + g_{lm}(S_j{}^r C_{ripk} + S_k{}^r C_{rijp} - S_p{}^r C_{rijk}) \\
& + g_{lj}(S_k{}^r C_{ripm} + S_m{}^r C_{rikp} - S_p{}^r C_{rikm}) \\
& + g_{lk}(S_m{}^r C_{ripj} + S_j{}^r C_{rimp} - S_p{}^r C_{rimj}) = 0.
\end{aligned}$$

Hence, by contraction with g^{lp} , we obtain (7). In local coordinates the relation (1) takes the form

$$\begin{aligned}
& C_{rijk}R_{hlm}^r + C_{hrjk}R_{ilm}^r + C_{hirk}R_{jlm}^r + C_{hijr}R_{klm}^r \\
& = L(S_{hl}C_{mijk} - S_{hm}C_{lijk} + S_{il}C_{hmjk} - S_{im}C_{hljk} + S_{jl}C_{himk} \\
& \quad - S_{jm}C_{hilk} + S_{kl}C_{hijm} - S_{km}C_{hijl}).
\end{aligned}$$

Contracting this equality with g^{hk} , in virtue of (7) and the assumption $L \neq 0$, we get $S_i{}^r C_{rjlm} + S_j{}^r C_{ril m} = 0$, i.e., (8). Applying (7) to (11) we find

$$\begin{aligned}
(12) \quad & 2(S_{mp}C_{likj} + S_{jp}C_{limk} + S_{kp}C_{lijm}) - (S_{lm} - \kappa g_{lm})C_{pijk} \\
& - (S_{lj} - \kappa g_{lj})C_{pikm} - (S_{lk} - \kappa g_{lk})C_{pimj} - (S_{im} - \kappa g_{im})C_{lpjk} \\
& - (S_{ij} - \kappa g_{ij})C_{lpkm} - (S_{ik} - \kappa g_{ik})C_{lpmj} \\
& + S_i{}^r(g_{lm}C_{rpjk} + g_{lj}C_{rpkm} + g_{lk}C_{rpmj}) \\
& + S_l{}^r(g_{im}C_{rpjk} + g_{ij}C_{rpmk} + g_{ik}C_{rpjm}) = 0.
\end{aligned}$$

Contracting (12) with g^{lm} , we obtain $2S_p{}^r C_{rijk} = S_j{}^r C_{rk ip} + S_k{}^r C_{rj pi}$ and next, in view of (8), we get (9). This completes the proof. $\square$

Proposition 4.1. *Let (M, g) , $\dim M = 4$, be a semi-Riemannian manifold satisfying the condition (1). If $L \neq 0$ at x , then the following relations are fulfilled at x :*

$$(13) \quad S(W, C(X, Y)Z) = \frac{\kappa}{4}C(X, Y, Z, W),$$

$$(14) \quad \begin{aligned} &T(U, X)C(W, V, Y, Z) + T(U, Y)C(W, V, Z, X) \\ &+ T(U, Z)C(W, V, X, Y) - T(V, X)C(W, U, Y, Z) \\ &- T(V, Y)C(W, U, Z, X) - T(V, Z)C(W, U, X, Y) = 0, \end{aligned}$$

where $T = S - \frac{\kappa}{4}g$ and $X, Y, Z, W, U, V \in T_x(M)$,

$$(15) \quad S^2 = \frac{\kappa}{2}S - \frac{\kappa^2}{16}g.$$

Moreover, (M, g) is Ricci-pseudosymmetric at x and

$$(16) \quad R \cdot S = \frac{\kappa}{12}Q(g, S).$$

PROOF. Transvecting the Patterson's identity with S^{hm} and using the equality $S^{rs}C_{rijs} = 0$, which is an obvious consequence of (8), we get

$$\kappa C_{lijk} = S_l^r C_{rijk} - S_i^r C_{rljk} - S_j^r C_{rkil} - S_k^r C_{rjli}.$$

This relation, in virtue of (9) takes the form $\kappa C_{lijk} = 2S_l^r C_{rijk} - 2S_i^r C_{rljk}$ whence, in view of (8), we obtain (13). Using (13) we have

$$\begin{aligned} &S_i^r (g_{lm} C_{rpjk} + g_{lj} C_{rpkm} + g_{lk} C_{rpmj}) + S_l^r (g_{im} C_{rpjk} + g_{ij} C_{rpmk} + g_{ik} C_{rpjm}) \\ &= \frac{\kappa}{4} (g_{lm} C_{ipjk} + g_{lj} C_{ipkm} + g_{lk} C_{ipmj} + g_{im} C_{lpkj} + g_{ij} C_{lpmk} + g_{ik} C_{lpjm}). \end{aligned}$$

Substituting this equality into (12) we obtain

$$(17) \quad \begin{aligned} &2(S_{mp} C_{likj} + S_{jp} C_{limk} + S_{kp} C_{lijm}) = \left(S_{lm} - \frac{3}{4} \kappa g_{lm} \right) C_{pijk} \\ &+ \left(S_{lj} - \frac{3}{4} \kappa g_{lj} \right) C_{pikm} + \left(S_{lk} - \frac{3}{4} \kappa g_{lk} \right) C_{pimj} \\ &+ \left(S_{im} - \frac{3}{4} \kappa g_{im} \right) C_{lpjk} + \left(S_{ij} - \frac{3}{4} \kappa g_{ij} \right) C_{lpkm} \\ &+ \left(S_{ik} - \frac{3}{4} \kappa g_{ik} \right) C_{lpmj}. \end{aligned}$$

On the other hand, transvecting the Patterson's identity with S_p^h and using (13) we get

$$S_{pm}C_{lijk} + S_{pj}C_{likm} + S_{pk}C_{limj} + \frac{\kappa}{4}(g_{im}C_{pljk} + g_{ij}C_{plkm} + g_{ik}C_{plmj} - g_{lm}C_{pijk} - g_{lj}C_{pikm} - g_{lk}C_{pimj}) = 0.$$

Substituting this equality into (17) we have (14).

Transvecting now (14) with T_h^m and using (13) we obtain $T_{hl}^2C_{pijk} = T_{hi}^2C_{pljk}$.

Applying now Lemma 2.1 for $A_{lpis_1s_2s_3} = T_{s_1l}^2C_{pis_2s_3}$, in virtue of $C \neq 0$, we get $T^2 = 0$. Thus we have (15). Finally, taking into account the equality (2), we have

$$S_l^r C_{rijk} = S_l^r R_{rijk} - \frac{1}{2}(S_{lk}S_{ij} - S_{lj}S_{ik}) - \frac{1}{2}(g_{ij}S_{lk}^2 - g_{ik}S_{lj}^2) + \frac{\kappa}{6}(S_{lk}g_{ij} - S_{lj}g_{ik}).$$

This equality, in virtue of (15) takes the form

$$S_l^r C_{rijk} = S_l^r R_{rijk} - \frac{1}{2}(S_{lk}S_{ij} - S_{lj}S_{ik}) - \frac{\kappa}{12}(g_{ij}S_{lk} - g_{ik}S_{lj}) + \frac{\kappa^2}{32}(g_{ij}g_{lk} - g_{ik}g_{lj}).$$

Symmetrizing this relation in i, l and using (8) we have

$$S_l^r R_{rijk} + S_i^r R_{rljk} = \frac{\kappa}{12}(g_{lj}S_{ik} - g_{lk}S_{ij} + g_{ij}S_{lk} - g_{ik}S_{lj}),$$

i.e., equality (16). This completes the proof. $\square$

Theorem 4.1. *Let (M, g) , $\dim M = 4$, be a semi-Riemannian manifold satisfying the condition (1). If $L \neq 0$ and $S \neq \frac{\kappa}{4}g$ at x , then the following relations hold at x :*

- (i) $\kappa \neq 0$,
- (ii) $\mathfrak{S}_{X,Y,Z} a(X)\mathcal{C}(Y, Z) = 0$ for a certain non-zero covector a at x ,
- (iii) $L = \frac{1}{3}$,
- (iv) $Q(T, C) = 0$.

Consequently, (M, g) is pseudosymmetric at x and $L_R = \frac{\kappa}{12}$.

PROOF. Since $T \neq 0$ at x , we may choose a vector V at x (with local components V^r), such that the scalar $\rho = a(V)$ is non-zero, where a is the covector defined by $a(X) = T(V, X)$. We also put $F_{ij} = V^r V^s C_{rij s}$, $E_{ijk} = V^r C_{rijk}$. Transvecting (14) with $V^l V^m$ we get

$$(18) \quad \rho C_{pijk} + a_j E_{kip} + a_k E_{jpi} + a_i E_{pjk} + T_{ij} F_{pk} - T_{ik} F_{pj} = 0,$$

which by transvection with V^p , leads to

$$(19) \quad E_{ijk} = \frac{1}{\rho} (a_k F_{ij} - a_j F_{ik}).$$

Symmetrizing (18) with respect to p, i we obtain

$$a_i E_{pjk} + a_p E_{ijk} + T_{ij} F_{pk} - T_{ik} F_{pj} + T_{pj} F_{ik} - T_{pk} F_{ij} = 0,$$

which, in view of (19), leads to $Q(F, T - \frac{1}{\rho} a \otimes a) = 0$. Applying now Lemma 2.2, in view of $F \neq 0$, we have $T - \frac{1}{\rho} a \otimes a = \omega F$, $\omega \in \mathbb{R}$.

We consider two cases: (I) $\omega = 0$ and (II) $\omega \neq 0$.

(I) $\omega = 0$. Then we have $T_{ij} = \frac{1}{\rho} a_i a_j$ and substituting this into (18) and using (19) we have

$$(20) \quad \rho^2 C_{pijk} = a_p a_k F_{ij} - a_p a_j F_{ik} + a_i a_j F_{pk} - a_i a_k F_{pj}.$$

This implies

$$(21) \quad a_l C_{pijk} + a_p C_{iljk} + a_i C_{lpjk} = 0$$

and, in view of Lemma 2.3 also $C \cdot C = 0$. Taking into account the relation (3) and using (13) and (1) we have ($n = 4$)

$$(22) \quad (1 - 2L)Q(S, C) = \frac{\kappa}{12}Q(g, C).$$

This implies $\kappa = 0$ if and only if $L = \frac{1}{2}$. We assert that $\kappa \neq 0$ at x . Suppose that $\kappa = 0$ at x . Then $T = S = \frac{1}{\rho} a \otimes a$ and using (20) we easily obtain $Q(S, C) = 0$, a contradiction. Now (22) implies $Q(S, C) = \frac{\kappa}{12(1-2L)}Q(g, C)$ whence, in virtue of (1), we have $R \cdot C = \frac{\kappa L}{12(1-2L)}Q(g, C)$. But (M, g) is Ricci-pseudosymmetric with $L_S = \frac{\kappa}{12}$. So we get $\frac{\kappa L}{12(1-2L)} = \frac{\kappa}{12}$ and we

have (iii). It is easy to see that for $L = \frac{1}{3}$ the equality (22) takes the form (iv).

Now we consider the second case:

(II) $\omega \neq 0$. Then

$$(23) \quad \begin{aligned} F_{ij} &= \frac{1}{\omega} \left(T_{ij} - \frac{1}{\rho} a_i a_j \right), \\ E_{ijk} &= \frac{1}{\rho \omega} (a_k T_{ij} - a_j T_{ik}). \end{aligned}$$

Substituting these relations into (18) we get

$$(24) \quad \begin{aligned} \rho^2 \omega C_{lijk} &= \rho (T_{ik} T_{lj} - T_{ij} T_{lk}) \\ &+ 2(a_l a_k T_{ij} - a_l a_j T_{ik} + a_i a_j T_{lk} - a_i a_k T_{lj}). \end{aligned}$$

Contracting (23) with g^{ij} we have $a_r a^r = 0$. Using this equality and (15), after contraction of (24) with g^{ij} , we get $a_i a^r T_{rj} + a_j a^r T_{ri} = 0$ which immediately implies $a^r T_{rj} = 0$. Transvecting now (24) with a^l and using the above equality we obtain $a^r C_{rij k} = 0$. This implies, by transvection of the Patterson's identity with a^h , $a_m C_{lijk} + a_j C_{likm} + a_k C_{limj} = 0$. Substituting (24) to this equality we have

$$(25) \quad a_m (T_{lk} T_{ij} - T_{lj} T_{ik}) + a_j (T_{lm} T_{ik} - T_{lk} T_{im}) + a_k (T_{lj} T_{im} - T_{lm} T_{ij}) = 0.$$

We assert that $\kappa \neq 0$ at x . Suppose that $\kappa = 0$ at x . Then $T = S$ and

$$\rho \omega C_{lijk} = (S_{ik} S_{lj} - S_{ij} S_{lk}) + \frac{2}{\rho} (a_l a_k S_{ij} - a_l a_j S_{ik} + a_i a_j S_{lk} - a_i a_k S_{lj}).$$

Using this relation and (25), after standard but somewhat lengthy calculations we obtain $Q(S, C) = 0$, a contradiction. The proof of (iii) and (iv) is the same as in the case (I). This completes the proof. $\square$

The existence of manifolds satisfying all relations of the above theorem can be established (see [10], Example 5.1) as follows

Example 4.2. Let $(N, \tilde{g})$, $\dim N = 3$, be a semi-Riemannian manifold such that its Ricci tensor $\tilde{S}$ is of rank one and its scalar curvature $\tilde{\kappa}$ vanishes identically on N . An example of such manifold is presented in [10] (Example 5.1 (ii)). Furthermore, let F , defined by $F(x^1) = a \exp(bx^1)$,

$a = \text{const.} \neq 0$, $b = \text{const.} \neq 0$, be a function on a 1-dimensional manifold $(\overline{M}, g_1)$. It is shown in [10] that warped product $\overline{M} \times_F N$ satisfies (1) with $L = \frac{1}{3}$ and is a pseudosymmetric manifold with $L_R = \frac{\kappa}{12}$, where κ is the scalar curvature of $\overline{M} \times_F N$. Moreover, it is easy to verify that $\kappa = -3b^2 \neq 0$ and one can see that the condition (ii) of the last theorem is satisfied at every point x of $\overline{M} \times_F N$ by arbitrary covector a at x such that the only non-zero component of a is a_2 .

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