

A counterexample concerning contractive projections of real JB*-triples

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Abstract. We describe the complete real polynomial vector fields of a Euclidean disc and we construct a contractive linear projection of a real JB*-triple onto a 2-dimensional subspace with Euclidean norm such that the projected triple product violates the Jordan identity.

1. Introduction

In 1982 the author established [9] that the image by a contractive linear projection of the unit ball of a complex Banach space is holomorphically symmetric whenever the unit ball itself has the same property. As a consequence of this fact, in 1984 KAUP proved [7] by the aid of his Riemann mapping theorem [6] on bounded symmetric domains that the image of a complex JB*-triple by a contractive linear projection is a JB*-triple with the projected product and this latter is the unique operation satisfying the JB*-triple axioms on the image space. This theorem answered positively a long standing conjecture stating that contractive linear images of complex C*-algebras are JB*-triples. Also this result gave rise to the possibility of generalizing the Arens product (defined originally for C*-algebras) to biduals of complex JB*-triples [3].

Recall that by a complex JB*-triple we mean a Banach space E equipped with an operation $\{xyz\}$ ($x, y, z \in E$) of three arguments (called

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the triple product) which is symmetric complex-bilinear in its outer variables x, z , conjugate-linear in the inner variable y , satisfies the C^* -axiom $\|\{xxx\}\| = \|x\|^3$ ($x \in E$), the Jordan identity $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ ($a, b, x, y, z \in E$), and the spectral axiom stating that, for any $a \in E$, the linear operator $D_a x := \{aax\}$ is E -Hermitian with non-negative spectrum (i.e. $\|\exp(\zeta D_a)\| \leq 1$ whenever $\operatorname{Re} \zeta \leq 0$). In particular complex C^* -algebras with the triple product $\{xyz\} := \frac{1}{2}xy^*z + \frac{1}{2}zy^*x$ can be regarded as complex JB^* -triples. Given a complex Banach space E , there can be defined a JB^* -triple product on E if and only if the unit ball is symmetric holomorphically and this product is uniquely determined in the latter case. Conversely, given an operation $\{ \} : E^3 \rightarrow E$ on a Banach space E , there exists at most one equivalent norm $\| \cdot \|$ on E (the so-called JB^* -norm of $\{ \}$) which makes $(E, \| \cdot \|, \{ \})$ a JB^* -triple. (For details see e.g. [11].)

Recently considerable efforts are paid to develop a theory of real JB^* -triples [1], [11], [5], [8] defined as real subspaces of complex JB^* -triples being closed under the underlying triple product. Some positive results [2], [4] have already appeared concerning the problem of contractive projections of real JB^* -triples, and several experts raise the conjecture that the contractive linear image of a real JB^* -triple is a real JB^* -triple with the projected product. The simple example of Section 2 in 4 real dimensions disproves this expectation: the projected product is no Jordan triple product on the range of a rank 2 contractive linear projection P of the realification of a 2 complex dimensional Cartan factor $(E, \| \cdot \|, \{ \})$ of Type 1. In our example the intersection D of the unit ball of E with the range of the projection P is a (2-dimensional) Euclidean disc. By the real version [10] of the projection principle, the vector fields of the form $P[c - \{xcx\}] \partial/\partial x \mid D$ are all complete in D .^{*} However, they do not constitute a Lie-triple with respect to the Lie triple product $[X(x) \partial/\partial x, Y(x) \partial/\partial x, Z(x) \partial/\partial x] := [[X(x) \partial/\partial x, Y(x) \partial/\partial x], Z(x) \partial/\partial x]$ where

$$(1.1) \quad [X(x) \partial/\partial x, Y(x) \partial/\partial x] := \lim_{\tau \downarrow 0} [X(x + \tau Y(x)) - Y(x + \tau X(x))] \partial/\partial x$$

^{*}In our context, given a function $f : E \rightarrow E$, we may identify $f(x) \partial/\partial x$ simply with f . The vector field $f(x) \partial/\partial x$ is said to be complete in D if for every $x_0 \in D$ there is a differentiable function $x : \mathbb{R} \rightarrow D$ such that $x(0) = x_0$ and $\frac{d}{dt}x(t) = f(x(t))$ ($t \in \mathbb{R}$).

is the usual Lie-commutator of vector fields. Our example based heuristically upon a complete parameterized list of the complete real polynomial vector fields on a (2-dimensional real) Euclidean disc, a result of independent interest which we discuss in Section 3. Among the underlying domains of real Cartan triple factors Hilbert balls play a distinguished role: their gauge functions can be the JB*-norm for several different real JB*-triple factors [8]. This latter fact seems to be one of the main obstacles on the way to a pure real geometric theory of JB*-triples, and it is commonly agreed that a deep understanding of the structure of the complete real polynomial vector fields of Hilbert balls can be crucial in this direction.

2. Counterexample

Proposition 2.1. *On the 2-dimensional complex space $\mathbb{C}^2$ let*

$$(2.2) \quad \{xyz\} := \frac{1}{2} \langle x | y \rangle z + \frac{1}{2} \langle z | y \rangle y \quad (x, y, z \in \mathbb{C}^2)$$

be the Jordan triple product of the complex type 1 Cartan factor structure of $\mathbb{C}^2$ with respect to the canonical scalar product $\langle x | y \rangle := x_1 \bar{y}_1 + x_2 \bar{y}_2$ and conjugation $\bar{x} := (\bar{x}_1, \bar{x}_2)$, and let P denote the real-linear projection

$$Px := \sum_{k=1}^2 \operatorname{Re} \langle x | e_k \rangle e_k \quad (x = (x_1, x_2) \in \mathbb{C}^2)$$

onto the real-linear subspace $\mathbb{R}e_1 + \mathbb{R}e_2$ with the unit vectors $e_1 := (1, 0)$, $e_2 := (i/\sqrt{2}, 1/\sqrt{2})$. Then the projection P is contractive with respect to the JB-triple norm $\|\cdot\|$ associated with (2.2) but the operation*

$$\{xyz\} := P\{xyz\} \quad (x, y, z \in \mathbb{R}e_1 + \mathbb{R}e_2)$$

violates the Jordan identity.

PROOF. It is well-known [8] that the JB*-triple norm of the triple product (2.2) on $\mathbb{C}^2$ coincides with the Hilbert norm associated with the scalar product, i.e.

$$\|x\| = \langle x | x \rangle^{1/2} = \left[\sum_{k=1}^2 (\operatorname{Re} \langle x | e_k \rangle)^2 \right]^{1/2} \quad (x \in \mathbb{C}^2)$$

where $e_3 := (-i/\sqrt{2}, 1/\sqrt{2})$ and $e_4 := (0, i)$. Since the system $\{e_1, e_2, e_3, e_4\}$ is orthonormed with respect to the real scalar product $\operatorname{Re} \langle x | y \rangle$ on $\mathbb{C}^2$, the operator P is an orthogonal projection with respect to $\operatorname{Re} \langle x | y \rangle$ and in particular contractive with respect to the norm $\| \cdot \|$. We have to show that

$$(2.3) \quad \{ab\}\{xyz\}_P\}_P \neq \{\{abx\}_P yz\}_P - \{x\{bay\}_P z\}_P + \{xy\{abz\}_P\}_P$$

for some $a, b, x, y, z \in \mathbb{R}e_1 + \mathbb{R}e_2$. For

$$(2.4) \quad a := e_2, \quad b := e_2, \quad x := e_2, \quad y := e_1, \quad z := e_2$$

we have inequality. Indeed

$$\begin{aligned} \{e_k e_k e_k\}_P &= P \langle e_k | e_k \rangle e_k = e_k \quad (k = 1, 2), \\ \{e_2 e_2 e_1\}_P &= \{e_1 e_2 e_2\}_P = \frac{1}{2} P [\langle e_2 | e_2 \rangle e_1 + \langle e_1 | e_2 \rangle e_2] \\ &= \frac{1}{2} P \left(e_1 - \frac{i}{\sqrt{2}} e_2 \right) = P \left(\frac{3}{4}, -\frac{i}{4} \right) = P \left(\frac{3}{4} e_1 - \frac{1}{4} e_4 \right) = \frac{3}{4} e_1, \\ \{e_2 e_1 e_2\}_P &= P [\langle e_2 | e_1 \rangle e_2] = P \left(\frac{i}{\sqrt{2}} e_2 \right) = P \left(-\frac{1}{2}, \frac{i}{2} \right) \\ &= P \left(-\frac{1}{2} e_1 + \frac{1}{2} e_4 \right) = -\frac{1}{2} e_1, \\ \{e_1 e_2 e_1\}_P &= P [\langle e_1 | e_2 \rangle e_1] = P \left(-\frac{i}{\sqrt{2}} e_1 \right) = P \left(-\frac{i}{\sqrt{2}}, 0 \right) \\ &= \frac{1}{2} P (e_3 - e_2) = -\frac{1}{2} e_2, \end{aligned}$$

It follows

$$\begin{aligned} \{ab\}\{xyz\}_P\}_P &= \{e_2 e_2 \{e_2 e_1 e_2\}_P\}_P = -\frac{1}{2} \{e_2 e_2 e_1\}_P = -\frac{3}{8} e_1, \\ \{\{abx\}_P yz\}_P &= \{\{e_2 e_2 e_2\}_P e_1 e_2\}_P = \{e_2 e_1 e_2\}_P = -\frac{1}{2} e_1, \\ \{x\{bay\}_P z\}_P &= \{e_2 \{e_2 e_2 e_1\}_P e_2\}_P = \frac{3}{4} \{e_2 e_1 e_2\}_P = -\frac{3}{8} e_1, \\ \{xy\{abz\}_P\}_P &= \{e_2 e_1 \{e_2 e_2 e_2\}_P\}_P = \{e_2 e_1 e_2\}_P = -\frac{1}{2} e_1. \end{aligned}$$

Therefore the left hand side in (2.3) equals $-3/8e_1$ while the right hand side takes the value $-5/8e_1$ for the choice (2.4). $\square$

Remark 2.5. It turns out from the above proof that $D := P\{x \in \mathbb{C}^2 : \|x\| < 1\} = \{\alpha_1 e_1 + \alpha_2 e_2 : \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1^2 + \alpha_2^2 < 1\}$ is a 2-dimensional Euclidean disc. Therefore there are even two different real Jordan triple products, namely

$$\begin{aligned} \{xyz\}_1 &:= \frac{1}{2} \operatorname{Re} \langle x | y \rangle z + \frac{1}{2} \operatorname{Re} \langle z | y \rangle x, \\ \{xyz\}_2 &:= \operatorname{Re} \langle x | y \rangle z + \operatorname{Re} \langle z | y \rangle x - \operatorname{Re} \langle x | \bar{z} \rangle \bar{y} \end{aligned}$$

which make $\operatorname{ran}(P)$ with the norm $\|\cdot\|$ a 2-dimensional real JB*-triple. That is the vector fields $[c - \{xcx\}_1] \partial/\partial x$ ($c \in \operatorname{ran}(P)$) resp. $[c - \{xcx\}_2] \partial/\partial x$ ($c \in \operatorname{ran}(P)$) are complete in D . Also all the polynomial vector fields $X_c := [c - \{xcx\}_P] \partial/\partial x$ ($c \in \operatorname{ran}(P)$) of degree 2 are complete in D . However, with the commutator of vector fields (1.1),

$$\{[X_a, [X_b, X_c]] : a, b, c \in \operatorname{ran}(P)\} \not\subset \{X_u : u \in \operatorname{ran}(P)\}.$$

3. Complete real polynomial vector fields on the disc

Throughout this section let x, y, z denote the coordinate functions

$$x : (\xi, \eta) \mapsto \xi, \quad y : (\xi, \eta) \mapsto \eta, \quad z := x + iy$$

on $\mathbb{R}^2$. Recall that by a polynomial P of the type $\mathbb{R}^2 \rightarrow \mathbb{R}$ of degree $\leq N$ we mean a function of the form $P = \sum_{\substack{k+\ell \leq N \\ k, \ell \geq 0}} \alpha_{k, \ell} x^k y^\ell$ with suitable real coefficients $\alpha_{k, \ell}$. Since $x = (z + \bar{z})/2$ and $y = i(\bar{z} - z)/2$, by induction on N it follows that $\mathbb{R}^2 \rightarrow \mathbb{R}$ polynomials of degree N can be written in the complex forms

$$P = \sum_{\substack{k+2\ell \leq N \\ k, \ell \geq 0}} |z|^{2\ell} [\mu_{k, \ell} z^k + \overline{\mu_{k, \ell}} \bar{z}^k] = \sum_{m=0}^N \left[p_m(|z|^2) z^m + \overline{p_m(|z|^2)} \bar{z}^m \right]$$

with suitable complex coefficients $\mu_{k, \ell}$ and some polynomials $p_0, \dots, p_N : \mathbb{R} \rightarrow \mathbb{C}$ (where each p_m is of degree $\leq (N-m)/2$). In particular P vanishes

at the points of the unit circle $\mathbb{T} := \{(\cos t, \sin t) : t \in \mathbb{R}\}$ if and only if $0 = P(\cos t, \sin t) = \sum_{m=0}^N \left[p_m(1)e^{imt} + \overline{p_m(1)}e^{-imt} \right]$ ($t \in \mathbb{R}$) which is equivalent to $p_m(1) = 0$ ($m = 0, \dots, N$). Since for a polynomial $p : \mathbb{R} \rightarrow \mathbb{C}$ we have $p(1) = 0$ iff $p(\rho) = (1 - \rho)q(\rho)$ for some polynomial q , we conclude that

$$(3.1) \quad \{P \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2) : p(\mathbb{T}) = 0\} = \{(1 - |z|^2)Q : Q \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2)\}.$$

In the sequel we identify $\mathbb{R}^2$ with $\mathbb{C}$ via the complex coordinate z . Thus we regard the point $(\xi, \eta) \in \mathbb{R}^2$ as the complex number $\xi + i\eta$ and the mapping $(\rho \cos \theta, \rho \sin \theta) \mapsto (\rho^m \cos m\theta, \rho^m \sin m\theta)$ is identified with the complex function z^m for $m = 0, 1, 2, \dots$. In terms of this identification we have the following description of the complete real polynomial vector fields of the unit disc $\mathbb{D} := \{(\xi, \eta) \in \mathbb{R}^2 : \xi^2 + \eta^2 < 1\} (\equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\})$.

Theorem 3.2. *Let $P \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2)$. Then the vector field $P(v) \partial/\partial v$ is complete in $\mathbb{D}$ if and only if P is a finite real linear combination of the functions*

$$\begin{aligned} iz, \quad \mu \bar{z}^m - \bar{\mu} z^{m+2} \quad & (\mu \in \mathbb{C}, \quad m = 0, 1, \dots), \\ (1 - |z|^2)Q \quad & (Q \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2) \equiv \text{Pol}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})). \end{aligned}$$

PROOF. Let $\mathcal{P}$ denote the set of all polynomials $P \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2)$ such that the vector field $P(v) \partial/\partial v$ is complete in $\mathbb{D}$. Since $\mathbb{D}$ is a (real-analytic) submanifold of $\mathbb{R}^2$ with the analytic boundary $\mathbb{T}$, for a polynomial $P \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2)$ we have $P \in \mathcal{P}$ if and only if P is tangent to the circle $\mathbb{T}$. That is,

$$(3.3) \quad \begin{aligned} \mathcal{P} &= \{P \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2) : P(\xi, \eta) \perp (\xi, \eta) \text{ for } \xi, \eta \in \mathbb{R}, \quad \xi^2 + \eta^2 = 1\} \\ &= \{P \in \text{Pol}(\mathbb{R}^2, \mathbb{R}^2) : \text{Re}(P(e^{i\tau})e^{-i\tau}) = 0 \quad (\tau \in \mathbb{R})\}. \end{aligned}$$

Let us write t for the natural coordinate function $t : \tau \mapsto \tau$ of the real line $\mathbb{R}$. Notice that, according to the identification $z : \mathbb{R}^2 \leftrightarrow \mathbb{C}$, $P(e^{it})$ is a complex valued trigonometric polynomial of degree N whenever P is a real polynomial $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ of degree N . Define

$$\begin{aligned} \mathcal{T} &:= \{\text{trigonometric polynomials } \mathbb{R} \rightarrow \mathbb{C}\} = \oplus_{k=-\infty}^{\infty} \mathbb{C}e^{ikt}, \\ \mathcal{S} &:= \{P(e^{it}) : P \in \mathcal{P}\} \end{aligned}$$

where $\oplus$ denotes algebraic direct sum. By (3.3) we have

$$(3.4) \quad \mathcal{S} = \{T \in \mathcal{T} : \operatorname{Re}(T \cdot e^{-it}) = 0\}.$$

Thus $\mathcal{S}$ is a real-linear subspace of $\mathcal{T}$. Given any $T \in \mathcal{S}$, by differentiating the relation $\operatorname{Re}(T \cdot e^{it}) = 0$ we see that also $0 = \operatorname{Re}(T' \cdot e^{it} - iT \cdot e^{it}) = \operatorname{Re}[(T' - iT)e^{it}]$. That is

$$A\mathcal{S} \subset \mathcal{S} \quad \text{where} \quad A(T) := T' - iT \quad (T \in \mathcal{T}).$$

Observe that the complex-linear operator A acts diagonally with imaginary eigenvalues over the canonical basis of $\mathcal{T}$:

$$Ae^{ikt} = i(k-1)e^{ikt} \quad (k = 0, \pm 1, \pm 2, \dots).$$

Since $\mathcal{S}$ is an A -invariant real-linear subspace of $\mathcal{T}$ and the eigenvalues of A^2 are real, namely $A^2e^{ikt} = -(k-1)^2e^{ikt}$, it follows

$$(3.5) \quad \mathcal{T} = \bigoplus_{m=0}^{\infty} \mathcal{T}_m \quad \text{where} \quad \mathcal{T}_m := \{T \in \mathcal{T} : A^2T = -m^2T\} \\ = \mathbb{C}e^{i(1+m)t} + \mathbb{C}e^{i(1-m)t},$$

$$\mathcal{S} = \bigoplus_{m=0}^{\infty} \mathcal{S}_m \quad \text{where} \quad \mathcal{S}_m := \mathcal{S} \cap \mathcal{T}_m.$$

Indeed, the decomposition $\mathcal{T} = \bigoplus_m \mathcal{T}_m$ is trivial; if $T \in \mathcal{S}$ then we can write $T = \sum_{m=0}^N T_m$ with suitable N and $T_m \in \mathcal{T}_m$ ($m = 0, \dots, N$) and here necessarily $T_m = \ell_m(A^2)T \in A^2\mathcal{S} \subset \mathcal{S}$ where ℓ_m is the Lagrange interpolation polynomial of degree N with the property $\ell_m(-k^2) = \delta_{mk}$ ($k = 0, \dots, N$). By (3.4) and (3.5),

$$\begin{aligned} \mathcal{S}_m &= \{T \in \mathcal{T}_m : \operatorname{Re}(T \cdot e^{-it}) = 0\} \\ &= \left\{ \sum_{\varepsilon=\pm 1} \mu_{\varepsilon} e^{i(\varepsilon m+1)t} : \operatorname{Re} \sum_{\varepsilon=\pm 1} \mu_{\varepsilon} e^{i\varepsilon m t} = 0 \right\} \\ &= \left\{ \sum_{\varepsilon=\pm 1} \mu_{\varepsilon} e^{i(1+\varepsilon m)t} : \operatorname{Re}[(\mu_1 + \overline{\mu_{-1}})e^{imt}] = 0 \right\} \\ &= \left\{ \sum_{\varepsilon=\pm 1} \mu_{\varepsilon} e^{i(1+\varepsilon m)t} : \mu_1 + \overline{\mu_{-1}} = 0 \right\} \\ &= \{\mu e^{i(1-m)t} - \overline{\mu} e^{i(1+m)t} : \mu \in \mathbb{C}\}. \end{aligned}$$

By setting $Z_{0,\mu} := (\mu - \bar{\mu})z$, $Z_{m,\mu} := \mu\bar{z}^{m-1} - \bar{\mu}z^{m+1}$ ($m > 0$, $\mu \in \mathbb{C}$), we have $\mu e^{i(1-m)t} - \bar{\mu} e^{i(1+m)t} = Z_{m,\mu}(e^{it})$. Thus for each real polynomial $P \in \mathcal{P}$ there exists some real linear combination of the real polynomials $Z_{m,\mu}$ which coincides with P on the boundary $\mathbb{T}$ of $\mathbb{D}$. That is, each element of $\mathcal{P}$ is the sum of some real polynomial vanishing on $\mathbb{T}$ with a real-linear combination of functions of the form $Z_{m,\mu}$. Taking (3.1) into account, this completes the proof. $\square$

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