

Functional equations in the theory of conditionally specified distributions

By KÁROLY LAJKÓ (Debrecen)

Abstract. The functional equation

$$G_1(xy + x) + F_1(y) = G_2(xy + y) + F_2(x)$$

related to the characterizations of bivariate distributions is investigated for functions $F_i, G_i : \mathbb{R} \rightarrow \mathbb{R}$ or $F_i, G_i : \mathbb{R}_+ \rightarrow \mathbb{R}$, respectively.

1. Introduction

Functional equations have many interesting applications in the characterization problems of probability theory.

In [1] ARNOLD, CASTILLO and SARABIA showed how solutions of functional equations can be used in characterizing joint distributions from conditional distributions and also an array of conditionally specified models was presented and analysed.

Let (X, Y) be an absolutely continuous bivariate random variable. Let us denote the joint, marginal and conditional densities by $f_{(X,Y)}$, f_X , f_Y , $f_{X|Y}$, $f_{Y|X}$, respectively. One can write $f_{(X,Y)}$ in two ways and obtain the functional equation

$$(1) \quad f_{X|Y}(x, y)f_Y(y) = f_{Y|X}(x, y)f_X(x)$$

Mathematics Subject Classification: 39B22.

Key words and phrases: characterizations of bivariate distributions, functional equations, general and measurable solutions.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. T-030082 and by the Hungarian High Educational Research and Development Found (FKFP) Grant No. 0310/1997.

for all $x, y \in \mathbb{R}$ (or for all $x, y \in \mathbb{R}_+$ if we restrict our search to the random variable (X, Y) with support in the positive quadrant).

For example, it is natural to inquire about the nature of all joint densities whose conditional densities satisfy

$$(2) \quad f_{X|Y}(x, y) = g_1((\alpha + y)x); \quad f_{Y|X}(x, y) = g_2((\beta + x)y)$$

for all $x, y \in \mathbb{R}$ or $x, y \in \mathbb{R}_+$, where $\alpha, \beta \in \mathbb{R}$ or $\alpha, \beta \in \mathbb{R}_+$ are arbitrary constants, respectively (see [1]). We ask for what functions g_1 and g_2 can we have (1) holding for $x, y \in \mathbb{R}$ or $x, y \in \mathbb{R}_+$, respectively.

From (1) and (2) we get that the functions $g_1, g_2, f_X, f_Y : \mathbb{R}$ (or $\mathbb{R}_+$) $\rightarrow \mathbb{R}_+$ satisfy the functional equation

$$(3) \quad g_1((\alpha + y)x)f_Y(y) = g_2((\beta + x)y)f_X(x)$$

for all $x, y \in \mathbb{R}$ (or for all $x, y \in \mathbb{R}_+$).

The solution of (3) can be reduced to the solution of the functional equation

$$(4) \quad G_1(xy + x) + F_1(y) = G_2(xy + y) + F_2(x)$$

($x, y \in \mathbb{R}$ or $x, y \in \mathbb{R}_+$) for functions $G_i, F_i : \mathbb{R}$ (or $\mathbb{R}_+$) $\rightarrow \mathbb{R}$.

In this paper, we present the general solution of (4) when the functions are defined on $\mathbb{R}$, and the measurable solution of (4) when all the functions are defined on $\mathbb{R}_+$.

2. The general solution of (4) on $\mathbb{R}$

Here we shall use the following result of D. BLANUŠA and Z. DARÓCZY (see [2], [3]).

Theorem B–D. *The function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the functional equation*

$$(H) \quad f(x + y - xy) + f(xy) = f(x) + f(y), \quad x, y \in \mathbb{R}$$

if and only if

$$(5) \quad f(x) = A(x) + b, \quad x \in \mathbb{R},$$

where A is an additive function on $\mathbb{R}^2$ and $b \in \mathbb{R}$ is an arbitrary constant.

Theorem 1. *The functions $F_i, G_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) satisfy the functional equation (4) for all $x, y \in \mathbb{R}$ if and only if*

$$(6) \quad F_i(x) = A(x) + b_i, \quad x \in \mathbb{R} \quad (i = 1, 2),$$

$$(7) \quad G_i(x) = A(x) + c_i, \quad x \in \mathbb{R} \quad (i = 1, 2),$$

where $A : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function on $\mathbb{R}^2$ and $b_i, c_i \in \mathbb{R}$ ($i = 1, 2$) are arbitrary constants with $b_1 + c_1 = b_2 + c_2$.

PROOF. Putting $x = 0$ or $y = 0$ or $x = 0, y = 0$ in (4) we get

$$(8) \quad G_1(0) + F_1(y) = G_2(y) + F_2(0), \quad y \in \mathbb{R},$$

and

$$(9) \quad G_1(x) + F_1(0) = G_2(0) + F_2(x), \quad x \in \mathbb{R},$$

and

$$(10) \quad G_1(0) + F_1(0) = G_2(0) + F_2(0)$$

respectively. Using these identities and (4) we have

$$(11) \quad G_1(xy + x) + F_1(y) = F_1(xy + y) + G_1(x), \quad x, y \in \mathbb{R}.$$

Putting $x = -1$ here we obtain

$$(12) \quad G_1(-y - 1) + F_1(y) = F_1(0) + G_1(-1), \quad y \in \mathbb{R}.$$

Substituting this into (11) we get the functional equation

$$G_1(xy + x) - G_1(-y - 1) = -G_1(-(xy + y) - 1) + G_1(x), \quad x, y \in \mathbb{R}.$$

Replacing here x, y by $-x, y - 1$, we get

$$G_1(-xy) + G_1(-(x + y - xy)) = G_1(-x) + G_1(-y), \quad x, y \in \mathbb{R},$$

which implies that the function f defined by

$$(13) \quad f(x) = G_1(-x), \quad x \in \mathbb{R}$$

satisfies the functional equation (H).

So, by Theorem B–D, f is of the form

$$(14) \quad f(x) = A_1(x) + c_1, \quad x \in \mathbb{R},$$

where $A_1 : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function on $\mathbb{R}^2$ and $c_1 \in \mathbb{R}$ is an arbitrary constant.

Taking (13) and (14) into consideration, we have (7) for G_1 with the additive function $A = -A_1$.

Then from (12), (9) and (8) we obtain (6) and (7) for the functions F_1, F_2 and G_2 , respectively, with real constants b_1, b_2, c_2 .

An easy calculation shows that the functions (6) and (7) indeed satisfy (4) if $b_1 + c_1 = b_2 + c_2$. $\square$

3. The general mesurable solution of (4) on $\mathbb{R}_+$

We need the following result of A. JÁRAI ([5] Theorem 2.7.2).

Theorem J. *Let $\mathcal{T}$ be a locally compact metric space, let Z_0 be a metric space, and let Z_i ($i = 1, 2, \dots, n$) be separable metric spaces. Suppose, that D is an open subset of $\mathcal{T} \times \mathbb{R}^k$ and $X_i \subset \mathbb{R}^k$ for $i = 1, 2, \dots, n$. Let $f_0 : \mathcal{T} \rightarrow Z_0$, $f_i : X_i \rightarrow Z_i$, $g_i : D \rightarrow X_i$, $H : D \times Z_1 \times Z_2 \times \dots \times Z_n \rightarrow Z_0$ be functions. Suppose, that the following conditions hold:*

(1) *For every $(t, y) \in D$*

$$f_0(t) = H(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))).$$

(2) *f_i is Lebesgue measurable over X_i for $i = 1, 2, \dots, n$.*

(3) *H is continuous on compact sets.*

(4) *For $i = 1, 2, \dots, n$, g_i is continuous, and for every fixed $t \in \mathcal{T}$ the mapping $y \rightarrow g_i(t, y)$ is differentiable with the derivative $D_2 g_i(t, y)$ and with the Jacobian $J_2 g_i(t, y)$, moreover, the mapping $(t, y) \rightarrow D_2 g_i(t, y)$ is continuous on D and for every $t \in \mathcal{T}$ there exists a $(t, y) \in D$ so that*

$$J_2 g_i(t, y) \neq 0 \quad \text{for } i = 1, 2, \dots, n.$$

Then f_0 is continuous on $\mathcal{T}$.

Lemma 1. *If the measurable functions $G_i, F_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($i = 1, 2$) satisfy the functional equation (4) for all $x, y \in \mathbb{R}_+$ then the functions G_i, F_i are continuous.*

PROOF. First we prove the continuity of G_1 . From (4), with $t = xy + x$, we obtain

$$(15) \quad G_1(t) = G_2\left(\frac{ty}{y+1} + y\right) + F_2\left(\frac{t}{y+1}\right) - F_1(y), \quad (t, y) \in \mathbb{R}_+^2.$$

Let $\mathcal{T} = \mathbb{R}_+$, $n = 3$, $Z_0 = Z_1 = Z_2 = Z_3 = \mathbb{R}$, $X_1, X_2, X_3 = \mathbb{R}_+$, $D = \mathbb{R}_+^2$. Define the functions g_i on $\mathbb{R}_+^2$ by

$$g_1(t, y) = \frac{ty}{y+1} + y, \quad g_2(t, y) = \frac{t}{y+1}, \quad g_3(t, y) = y$$

and let $H(t, y, z_1, z_2, z_3) = z_1 + z_2 - z_3$.

It follows from (15) that the functions f_i ($i = 1, 2, 3$) given by

$$f_0 = G_1, \quad f_1 = G_2, \quad f_2 = F_2, \quad f_3 = F_1$$

satisfy the functional equation in (1) of Theorem J for all $t, y \in D = \mathbb{R}_+^2$ and f_i ($i = 0, 1, 2, 3$) is measurable by the conditions of our lemma. H is continuous and condition (4) of Theorem J holds, too, since

$$D_2 g_1(t, y) = \frac{t}{(y+1)^2} + 1 \neq 0, \quad D_2 g_2(t, y) = -\frac{t}{(y+1)^2} \neq 0, \\ D_2 g_3(t, y) = 1 \neq 0$$

for all $(t, y) \in D = \mathbb{R}_+^2$.

Thus, by Theorem J, $f_0 = G_1$ is continuous on $\mathbb{R}_+$. The continuity of G_2 can be proved by making the substitutions $x \rightarrow y$, $y \rightarrow x$ in (4) and repeating the above argument.

Putting $x = 1$ or $y = 1$ in (4) and solving the equation obtained for F_1 and F_2 , respectively, we get

$$(16) \quad F_1(y) = G_2(2y) - G_1(y+1) + F_2(1), \quad y \in \mathbb{R}_+,$$

and

$$(17) \quad F_2(x) = G_1(2x) - G_2(x+1) + F_1(1), \quad x \in \mathbb{R}_+,$$

respectively. Whence by the continuity of G_1, G_2 it follows that F_1 and F_2 are continuous as well. $\square$

Lemma 2. *If the measurable functions $G_i, F_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($i = 1, 2$) satisfy the functional equation (4) for all $x, y \in \mathbb{R}_+$ then they are differentiable infinitely many times on $\mathbb{R}_+$.*

PROOF. Write (4) in the form (15) and let $[\alpha, \beta] \subset \mathbb{R}_+$ be arbitrary and choose the interval $[\lambda, \mu] \subset \mathbb{R}_+$ arbitrarily, too, then $[\alpha, \beta] \times [\lambda, \mu] \subset D = \mathbb{R}_+^2$ holds.

Integrating (15) with respect to y on $[\lambda, \mu]$ we obtain

$$(\mu - \lambda)G_1(t) = \int_{\lambda}^{\mu} G_2\left(\frac{ty}{y+1} + y\right) dy + \int_{\lambda}^{\mu} F_2\left(\frac{t}{y+1}\right) dy - \int_{\lambda}^{\mu} F_1(y) dy.$$

We use the substitutions

$$g_1(t, y) = \frac{ty}{y+1} + y = u, \quad g_2(t, y) = \frac{t}{y+1} = u$$

in the first and second integral, respectively. It is easy to check that these equations can uniquely be solved for y if $t \in [\alpha, \beta]$.

In the case $\frac{t}{y+1} = u$ this is clear. In the case $\frac{ty}{1+y} + y = u$ this uniqueness is ensured, namely the derivative of the function $y \rightarrow g_1(t, y)$:

$$D_2 g_1(t, y) = \frac{t}{(y+1)^2} + 1$$

is positive on $[\alpha, \beta] \times [\lambda, \mu]$, hence our function is strictly increasing.

The solutions

$$y = \frac{-(t-u+1) + \sqrt{(t-u+1)^2 + 4u}}{2} =: \gamma_1(t, u), \quad y = \frac{t}{u} - 1 =: \gamma_2(t, u)$$

are infinitely many times differentiable functions of t and u . Performing the substitutions we have

$$G_1(t) = \frac{1}{\mu - \lambda} \left[\int_{\frac{\lambda t}{\lambda+1} + \lambda}^{\frac{\mu t}{\mu+1} + \mu} G_2(u) D_2 \gamma_1(t, u) du + \int_{\frac{t}{\lambda+1}}^{\frac{t}{\mu+1}} F_2(u) D_2 \gamma_2(t, u) du - C \right],$$

where $C = \int_{\lambda}^{\mu} F_1(y) dy$. The functions G_2, F_2 are at least continuous. Hence, by repeated application of the theorem concerning the differentiable

of parametric integrals (see e.g. [4]), the right hand side is differentiable infinitely many times on $[\alpha, \beta]$. Since $[\alpha, \beta]$ is an arbitrary subinterval of $\mathbb{R}_+$, we have that G_1 is differentiable infinitely many times on $\mathbb{R}_+$. The differentiability of G_2 can be obtained similarly.

Finally from (16) and (17) we can deduce that F_1 and F_2 are also differentiable infinitely many times on $\mathbb{R}_+$. $\square$

Lemma 3. *If the functions $G_i, F_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($i = 1, 2$) satisfy the functional equation (4) for all $x, y \in \mathbb{R}_+$ and they are twice differentiable in $\mathbb{R}_+$, then there exist constants $C, \gamma, \delta_i \in \mathbb{R}$ ($i = 1, 2, 3, 4$), with $\delta_1 + \delta_3 = \delta_2 + \delta_4$ such that*

$$(18) \quad G_1(x) = C \ln x + \gamma x + \delta_1, \quad x \in \mathbb{R}_+,$$

$$(19) \quad F_1(x) = C \ln \frac{x}{x+1} + \gamma x + \delta_3, \quad x \in \mathbb{R}_+,$$

$$(20) \quad G_2(x) = C \ln x + \gamma x + \delta_2, \quad x \in \mathbb{R}_+,$$

$$(21) \quad F_2(x) = C \ln \frac{x}{x+1} + \gamma x + \delta_4, \quad x \in \mathbb{R}_+.$$

PROOF. Differentiating (4) with respect to x , then the resulting equation with respect to y , we have

$$G'_1(xy+x) + (xy+x)G''_1(xy+x) = G'_2(xy+y) + (xy+y)G''_2(xy+y),$$

$$x, y \in \mathbb{R}_+.$$

This can hold if and only if

$$tG''_1(t) + G'_1(t) = \gamma = sG''_2(s) + G'_2(s), \quad t, s \in \mathbb{R}_+$$

for some constant γ .

The general solutions of the differential equations

$$tG''_1(t) + G'_1(t) = \gamma, \quad t \in \mathbb{R}_+,$$

and

$$sG''_2(s) + G'_2(s) = \gamma, \quad s \in \mathbb{R}_+,$$

have the following forms

$$G_1(t) = C \ln t + \gamma t + \delta_1, \quad t \in \mathbb{R}_+,$$

$$G_2(s) = C \ln s + \gamma s + \delta_2, \quad s \in \mathbb{R}_+,$$

where $C, \gamma, \delta_1, \delta_2 \in \mathbb{R}$ are arbitrary constants, thus G_1 and G_2 are of the forms (18) and (20), respectively. Then, from (16), (17), (18) and (20), we get (19) and (21) for F_1 and F_2 , respectively.

It is easy to see that (18), (19), (20) and (21) satisfy (4) if $\delta_1 + \delta_3 = \delta_2 + \delta_4$. $\square$

We may sum up the results of Lemmas 1, 2, 3 in the following theorem.

Theorem 2. *If the measurable functions $G_i, F_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($i = 1, 2$) satisfy the functional equation (4) for all $x, y \in \mathbb{R}_+$, then there exist constants $C, \gamma, \delta_i \in \mathbb{R}$ ($i = 1, 2, 3, 4$) such that G_1, F_1, G_2 and F_2 have the forms (18), (19), (20) and (21), respectively and $\delta_1 + \delta_3 = \delta_2 + \delta_4$.*

References

- [1] B. C. ARNOLD, E. CASTILLO and J. M. SARABIA, Conditionally Specified Distributions, Lecture Notes in Statistics 73, *Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Hong Kong-Barcelona-Budapest*, 1992.
- [2] D. BLANUŠA, The functional equation $f(x + y - xy) + f(xy) = f(x) + f(y)$, *Aequationes Math.* **5** (1970), 63–70.
- [3] Z. DARÓCZY, On the general solution of the functional equation $f(x + y - xy) + f(xy) = f(x) + f(y)$, *Aequationes Math.* **6** (1971), 130–132.
- [4] J. DIEUDONNÉ, Grundzüge der modernen Analysis, *VEB Deutscher Verlag der Wissenschaften, Berlin*, 1971.
- [5] A. JÁRAI, On measurable solutions of functional equations, *Publ. Math. Debrecen* **26** (1979), 17–35.

LAJKÓ KÁROLY
 INSTITUTE OF MATHEMATICS AND INFORMATICS
 LAJOS KOSSUTH UNIVERSITY
 H-4010 DEBRECEN, P.O. BOX 12
 HUNGARY
E-mail: lajko@math.klte.hu

(Received December 21, 1999; revised October 31, 2000)