

On R-quadratic Finsler spaces

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Abstract. In this paper, we introduce the notion of R-quadratic Finsler metric. It is known that every Berwald metric is R-quadratic and Landsbergian. We show that every compact R-quadratic Finsler space must be Landsbergian.

1. Introduction

In Finsler geometry, there are several notions of curvatures. Among them, the Riemann curvature is an important quantity. For a Finsler space (M, F) , the Riemann curvature is a family of linear transformations $\mathbf{R}_y : T_x M \rightarrow T_x M$, where $y \in T_x M$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$ (the definition will be given in S2). If F is Riemannian, i.e., $F(y) = \sqrt{g(y, y)}$ for some Riemannian metric g , then $\mathbf{R}_y := \mathbf{R}(\cdot, y)y$, where $\mathbf{R}(u, v)z$ denotes the Riemannian curvature tensor of g . In this case, $\mathbf{R}_y$ is quadratic in $y \in T_x M$. A Finsler metric is said to be *R-quadratic* if its Riemann curvature $\mathbf{R}_y$ is quadratic in $y \in T_x M$. There are many non-Riemannian R-quadratic Finsler metrics. For example, all Berwald metrics are R-quadratic. Thus R-quadratic Finsler spaces form a rich class of Finsler spaces. The main purpose of this paper is to prove the following

Theorem 1.1. *Let (M, F) be a positively complete Finsler space with bounded Cartan torsion. Suppose that F is R-quadratic, then F must be a*

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Landsberg metric. In particular, every compact R-quadratic Finsler space must be Landsbergian.

The second part of Theorem 1.1 is true because that Finsler metrics on a compact manifold must be positively complete with bounded Cartan torsion. Theorem 1.1 tells us that for Finsler metrics on a compact manifold, the following holds

$$\{\text{Berwald metrics}\} \subset \{\text{R-quadratic metrics}\} \subset \{\text{Landsberg metrics}\}.$$

It is an open problem in Finsler geometry whether or not there is a Landsberg metric which is not Berwaldian. Theorem 1.1 throws a light into this problem.

We will see that a Finsler metric is R-quadratic if and only if the h -curvature of the Berwald connection depends on position only in the sense of Bácsó–Matsumoto (see Remark 3.1 below). In [BM], BÁCSÓ and MATSUMOTO classify Finsler metrics in the form $F(y) = \sqrt{g(y, y)} + \beta(y)$ (Randers metrics) whose h -curvature depend on position only. Their results indicates that there are possibly local R-quadratic Finsler metrics which are not Landsbergian.

For a submanifold M in a Minkowski space (V, F) , the Cartan torsion must be bounded [Sh1]. We obtain the following

Corollary 1.2. *For any positively complete submanifold M in a Minkowski space (V, F) , if the induced Finsler metric $\bar{F}$ is R-quadratic, then $\bar{F}$ must be a Landsberg metric.*

A Finsler space is said to be *R-flat*, if the Riemann curvature $\mathbf{R}_y = 0$. R-flat Finsler metrics are of course R-quadratic. According to AKBAR-ZEDAH [AZ], for positively complete R-flat Finsler space (M, F) , if the Cartan torsion and its vertical covariant derivative are bounded, then F is locally Minkowski. The conditions on the Cartan torsion are satisfied by submanifolds in a Minkowski space. We conclude that for a positively complete submanifold in a Minkowski space, if the induced Finsler metric is R-flat, then it must be locally Minkowski.

2. Preliminaries

A Finsler metric on a manifold M is a nonnegative function F on TM having the following properties

- (a) F is C^∞ on $TM \setminus \{0\}$;
- (b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;
- (c) for each $y \in T_x M$, the following quadratic form $\mathbf{g}_y$ on $T_x M$ is positive definite,

$$(1) \quad \mathbf{g}_y(u, v) := \frac{1}{2} [F^2(y + su + tv)]|_{s,t=0}, \quad u, v \in T_x M.$$

At each point $x \in M$, $F_x := F|_{T_x M}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_x M \setminus \{0\}$. To measure the non-Euclidean feature of F_x , define $\mathbf{C}_y : T_x M \times T_x M \times T_x M \rightarrow R$ by

$$(2) \quad \mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [\mathbf{g}_{y+tw}(u, v)]|_{t=0}, \quad u, v, w \in T_x M.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM \setminus \{0\}}$ is called the *Cartan torsion*. E. Cartan got this quantity when he introduced his metric-compatible connection. Since then, it is called the Cartan tensor or the Cartan torsion in literatures.

F is said to be *positively complete* if every geodesic on (a, b) can be extended to a geodesic on (a, ∞) . A curve $c(t)$ is called a *geodesic* if it satisfies

$$(3) \quad \frac{d^2 c^i}{dt^2}(t) + 2G^i(\dot{c}(t)) = 0,$$

where $G^i(y)$ are local functions on TM given by

$$(4) \quad G^i(y) := \frac{1}{4} g^{il}(y) \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l}(y) y^k - \frac{\partial [F^2]}{\partial x^l}(y) \right\}, \quad y \in T_x M.$$

F is called a *Berwald metric* if $G^i(y)$ are quadratic in $y \in T_x M$ for all $x \in M$.

The Riemann curvature can be defined using geodesic fields and the induced Riemannian metrics. A local vector field Y is called a *geodesic field* if the integral curves of Y are geodesics. Fix a vector $y \in T_x M \setminus \{0\}$ and

extend it to a local geodesic field Y on a neighborhood U_x of x . Y defines a Riemannian metric on U_x by

$$\hat{g}_z(u, v) := g_{ij}(Y_z)u^i v^j, \quad u, v \in T_z U.$$

Let $\hat{R}(u, v)z$ denote the Riemannian curvature tensor of $\hat{g}$. Define $\mathbf{R}_y : T_x M \rightarrow T_x M$ by

$$(5) \quad \mathbf{R}_y(u) := \hat{R}(u, y)y, \quad u \in T_x M.$$

$\mathbf{R}_y$ is a well-defined linear transformation independent of the geodesic extension Y of y . The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM \setminus \{0\}}$ is called the *Riemann curvature*. The notion of Riemann curvature was first extended to Finsler metrics by L. BERWALD from a different approach [Bw]. F is said to be *R-quadratic* if $\mathbf{R}_y$ is quadratic in $y \in T_x M$ at each point $x \in M$. Berwald metrics are always R-quadratic (see Remark 3.1).

Let $U(t)$ be a vector field along a curve $c(t)$. The canonical covariant derivative $D_{\dot{c}}U(t)$ is defined by

$$(6) \quad D_{\dot{c}}U(t) := \left\{ \frac{dU^i}{dt}(t) + U^j(t) \frac{\partial G^i}{\partial y^j}(\dot{c}(t)) \right\} \frac{\partial}{\partial x^i} \Big|_{c(t)}.$$

$U(t)$ is said to be *parallel* along c if $D_{\dot{c}(t)}U(t) = 0$.

To measure the changes of the Cartan torsion $\mathbf{C}$ along geodesics, we define $\mathbf{L}_y : T_x M \times T_x M \times T_x M \rightarrow R$ by

$$(7) \quad \mathbf{L}_y(u, v, w) := \frac{d}{dt} [\mathbf{C}_{\dot{c}(t)}(U(t), V(t), W(t))] \Big|_{t=0}$$

where $c(t)$ is a geodesic and $U(t)$, $V(t)$, $W(t)$ are parallel vector fields along $c(t)$ with $\dot{c}(0) = y$, $U(0) = u$, $V(0) = v$, $W(0) = w$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM \setminus \{0\}}$ is called the *Landsberg curvature*. A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$. An important fact is that if F is Berwaldian, then it is Landsbergian. See Remark 3.3.

3. Structure equations

To find the relationship among various quantities, usually we go to the slit tangent bundle $\pi : \mathcal{T}M := TM \setminus \{0\} \rightarrow M$ and choose a linear

connection (such as the Berwald connection, the Cartan connection and the Chern connection) on an appropriate vector bundle (such as the vertical tangent bundle $\mathcal{V}TM$ and the pull-back tangent bundle π^*TM). In this paper, we will study them using differential forms on TM instead.

Let F be a Finsler metric on an n -dimensional manifold M . Let (x^i, y^i) be a standard coordinate system in TM and $G^i(y)$ denote the geodesic coefficients of F in (4). Put

$$(8) \quad \omega_j^i := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y) dx^k.$$

In literatures, $\frac{\partial^2 G^i}{\partial y^j \partial y^k}$ are denoted by G_{jk}^i . $\{\omega_j^i\}$ are called the *Berwald connection forms*. Put

$$(9) \quad g_{ij}(y) := \mathbf{g}_y(e_i, e_j),$$

$$(10) \quad C_{ijk}(y) := \mathbf{C}_y(e_i, e_j, e_k), \quad L_{ijk}(y) := \mathbf{L}_y(e_i, e_j, e_k).$$

where $\{e_i = \frac{\partial}{\partial x^i}|_{\pi(y)}\}$ is a natural local frame on M . They are local functions on TM . With the Berwald connection forms, we define $C_{ijk;l}$ and $C_{ijk;l}$ by

$$(11) \quad dC_{ijk} - C_{pj k} \omega_i^p - C_{ip k} \omega_j^p - C_{ij p} \omega_k^p = C_{ijk;l} \omega^l + C_{ijk;l} \omega^{n+l}.$$

The definition of $\mathbf{L}$ in (7) is equivalent to the following

$$(12) \quad L_{ijk}(y) := C_{ijk;l}(y) y^l.$$

In literatures, $C_{ijk;l}(y) y^l$ are also denoted by $C_{ijk|0}(y)$. Thus $L_{ijk} = C_{ijk|0}$.

Let $\omega^i := dx^i$ and $\omega^{n+i} := dy^i + y^j \omega_j^i$. $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a natural coframe for $T^*(TM)$. They satisfy the following structure equations

$$(13) \quad d\omega^i = \omega^j \wedge \omega_j^i,$$

$$(14) \quad dg_{ij} - g_{kj} \omega_i^k - g_{ik} \omega_j^k = -2L_{ijk} \omega^k + 2C_{ijk} \omega^{n+k},$$

See [Sh2] for a proof.

The Riemann curvature $\mathbf{R}_y$ defined in (5) gives rise to a set of local function R_k^i on TM

$$(15) \quad \mathbf{R}_y(e_k) =: R_k^i(y) \frac{\partial}{\partial x^i} \big|_x, \quad u \in T_x M.$$

In local coordinates, $R_k^i(y)$ can be expressed in terms of $G^i(y)$

$$(16) \quad R_k^i(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

See [Sh2]. Thus the Riemann curvature defined in (5) coincides with the usual one defined in a different way [Bw].

Remark 3.1. If F is Berwaldian, i.e., $G^i(y)$ are quadratic in $y \in T_x M$, then $R_k^i(y)$ are quadratic in $y \in T_x M$. Put

$$R_j^i{}_{kl}(y) := \frac{1}{3} \frac{\partial}{\partial y^j} \left\{ \frac{\partial R_k^i}{\partial y^l} - \frac{\partial R_l^i}{\partial y^k} \right\}.$$

$R_j^i{}_{kl}$ are the coefficients of the h -curvature of the Berwald connection, which are also denoted by $H_j^i{}_{kl}$ in literatures. We have

$$R_k^i(y) = y^j R_j^i{}_{kl}(y) y^l.$$

Thus $R_k^i(y)$ is quadratic in $y \in T_x M$ if and only if $R_j^i{}_{kl}(y)$ are functions of x only.

There is another set of local functions B_{jkl}^i on $\mathcal{T}M$ defined by

$$B_{jkl}^i(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y).$$

B_{jkl}^i are also denoted by G_{jkl}^i in literatures. Because this quantity was introduced by L. Berwald first, I call it the Berwald curvature in my papers. Note that F is Berwaldian if and only if $B_{jkl}^i = 0$.

We have

$$(18) \quad \Omega_j^i := d\omega_j^i - \omega_j^k \wedge \omega_k^i$$

$$(19) \quad = \frac{1}{2} R_j^i{}_{kl} \omega^k \wedge \omega^l - B_{jkl}^i \omega^k \wedge \omega^{n+l}.$$

The following lemma is crucial in our proof of the main result.

Lemma 3.2.

$$(20) \quad L_{ijk;l} y^l = \frac{1}{2} y^s y^l g_{ps} R_i^p{}_{kl;j}.$$

PROOF. Define $g_{ij;k}$ and $g_{ij\cdot k}$ by

$$dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = g_{ij;k}\omega^k + g_{ij\cdot k}\omega^{n+k}.$$

(14) means

$$(21) \quad g_{ij;k} = -2L_{ijk}, \quad g_{ij\cdot k} = 2C_{ijk}.$$

Define $y_{\cdot k}^i$ and $y_{\cdot k}^i$ by

$$\omega^{n+i} = dy^i + y^j\omega_j^i = y_{\cdot k}^i\omega^k + y_{\cdot k}^i\omega^{n+k}.$$

This means

$$(22) \quad y_{\cdot k}^i = 0, \quad y_{\cdot k}^i = \delta_k^i.$$

Differentiating (14) yields the following Ricci identities.

$$(23) \quad g_{pj}\Omega_i^p + g_{ip}\Omega_j^p = -2L_{ijk;l}\omega^k \wedge \omega^l - 2L_{ijk\cdot l}\omega^k \wedge \omega^{n+l} \\ - 2C_{ijl;k}\omega^k \wedge \omega^{n+l} - 2C_{ijl\cdot k}\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijp}\Omega_l^p y^l.$$

It follows from (23) that

$$(24) \quad C_{ijl;k} + L_{ijk\cdot l} = \frac{1}{2}g_{pj}B_{ikl}^p + \frac{1}{2}g_{ip}B_{jkl}^p.$$

Contracting (24) with y^j and using (22) yield

$$(25) \quad L_{jkl} = -\frac{1}{2}y^m g_{im} B_{jkl}^i.$$

Remark 3.3. F is Berwaldian if and only if $B_{jkl}^i = 0$. Thus Berwald metrics are always Landsbergian.

Contracting (24) with y^k yields (12). Differentiating (18) yields

$$(26) \quad d\Omega_j^i = -\Omega_j^k \wedge \omega_k^i + \omega_j^k \wedge \Omega_k^i.$$

Define $R_j^i{}_{kl;m}$ and $R_j^i{}_{kl\cdot m}$ by

$$(27) \quad dR_j^i{}_{kl} - R_m^i{}_{kl}\omega_i^m - R_j^i{}_{ml}\omega_k^m - R_j^i{}_{km}\omega_l^m + R_j^m{}_{kl}\omega_m^i \\ =: R_j^i{}_{kl;m}\omega^m + R_j^i{}_{kl\cdot m}\omega^{n+m}.$$

Similarly, we define $B_{jkl;m}^i$ and $B_{jkl\cdot m}^i$. From (26), one obtains the following Bianchi identity

$$(28) \quad R_j^i{}_{kl\cdot m} = B_{jml;k}^i - B_{jkm;l}^i.$$

Contracting (28) with $y^s g_{is}$ and using (25) yield

$$(29) \quad L_{jkm;l} - L_{jml;k} = \frac{1}{2} y^s g_{ps} R_j^p{}_{kl\cdot m}.$$

Contracting (29) with y^l yields (20). $\square$

4. Proof of Theorem 1.1

Let (M, F) be a Finsler space and $c : [a, b] \rightarrow M$ a geodesic. For a parallel vector field $V(t)$ along c ,

$$(30) \quad \mathbf{g}_{\dot{c}(t)}(V(t), V(t)) = \text{constant}.$$

Lemma 4.1. *Let (M, F) be a Finsler space. Suppose that F is R-quadratic. Then for any geodesic $c(t)$ and any parallel vector field $V(t)$ along c , the following functions*

$$(31) \quad \mathbf{C}(t) := \mathbf{C}_{\dot{c}}(V(t), V(t), V(t))$$

must be in the following forms

$$(32) \quad \mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0).$$

PROOF. By assumption, $R_j^i{}_{kl}(y)$ are functions of x only. Thus

$$R_j^i{}_{kl\cdot m} = \frac{\partial R_j^i{}_{kl}}{\partial y^m} = 0.$$

It follows from (20) that

$$(33) \quad L_{ijk;l} y^l = 0.$$

Let

$$(34) \quad \mathbf{L}(t) := \mathbf{L}_{\dot{c}}(V(t), V(t), V(t)).$$

From our definition of $\mathbf{L}_y$, we have

$$\mathbf{L}(t) = \mathbf{C}'(t).$$

By (33), we obtain

$$(35) \quad \mathbf{L}'(t) = L_{ijk;l}(\dot{c}(t))\dot{c}^l(t)V^i(t)V^j(t)V^k(t) = 0.$$

Then (32) follows. $\square$

To prove Theorem 1.1, take an arbitrary unit vector $y \in T_x M$ and an arbitrary vector $v \in T_x M$. Let $c(t)$ be the geodesic with $\dot{c}(0) = y$ and $V(t)$ the parallel vector field along c with $V(0) = v$. Define $\mathbf{C}(t)$ and $\mathbf{L}(t)$ as in (31) and (34), respectively. Then

$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0).$$

Suppose that $\mathbf{C}_y$ is bounded, i.e., there is a constant $K < \infty$ such that

$$\|\mathbf{C}\|_x := \sup_{y \in T_x M \setminus \{0\}} \sup_{v \in T_x M} \frac{|\mathbf{C}_y(v, v, v)|}{[\mathbf{g}_y(v, v)]^{\frac{3}{2}}} \leq K.$$

By (30), we know that

$$Q := \mathbf{g}_{\dot{c}(t)}(V(t), V(t))$$

is a positive constant. Thus

$$|\mathbf{C}(t)| \leq KQ^{\frac{3}{2}} < \infty.$$

and $\mathbf{C}(t)$ is a bounded function on $[0, \infty)$. This implies

$$\mathbf{L}_y(v, v, v) = \mathbf{L}(0) = 0.$$

Therefore $\mathbf{L} = 0$ and F is a Landsberg metric. This completes the proof of Theorem 1.1.

Corollary 4.2. *For any positively complete Randers metric $F = \alpha + \beta$ on a manifold M , if F is R-quadratic, then it must be a Berwald space.*

PROOF. First we know that the Cartan torsion of F must be bounded. In fact, $\|\mathbf{C}\|_x \leq 3/\sqrt{2}$ for any $x \in M$ (see Appendix below). By Theorem 1.1, F is a Landsberg metric. In a 1974 paper [M], MATSUMOTO showed that $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel. In a 1977 paper [HI], M. HASHIGUCHI and Y. ICHIJYŌ showed that for a Randers metric $F = \alpha + \beta$, if β is parallel, then F is a Berwald metric. This completes the proof. $\square$

5. Appendix

In this section, we will prove the following

Proposition 5.1. *For any Randers norm $F = \alpha + \beta$ in an n -dimensional vector space V with $\|\beta\| := \sup_{\alpha(y)=1} \beta(y) < 1$, the Cartan torsion satisfies*

$$(36) \quad \|\mathbf{C}\| \leq \frac{3}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \|\beta\|^2}}.$$

Proposition 5.1 in dimension two is proved in Exercise 11.2.6 in [BCS]. We will first give a different argument in dimension two, then extend it to higher dimensions.

Assume that $\dim V = 2$. The unit circle $S = F^{-1}(1)$ is a simple closed curve around the origin. For a unit vector $y \in S$, there is a vector $y^\perp \in V$ satisfying

$$(37) \quad g_y(y, y^\perp) = 0, \quad g_y(y^\perp, y^\perp) = 1.$$

The set $\{y, y^\perp\}$ is called the *Berwald basis* at y . Define

$$\mathbf{I}(y) := \mathbf{C}_y(y^\perp, y^\perp, y^\perp), \quad y \in S.$$

We call $\mathbf{I}$ the *main scalar*. Note that $\mathbf{I} = 0$ if and only if $\mathbf{C} = 0$. Moreover,

$$\|\mathbf{C}\| = \sup_{y \in S} |\mathbf{I}(y)|.$$

Fix a basis $\{e_1, e_2\}$ for V . Parameterize S by a counter-clockwise map $c(t) = u(t)e_1 + v(t)e_2$. Then

$$(38) \quad \sigma(t) := g_{c(t)}(\dot{c}(t), \dot{c}(t)) = \frac{u'(t)v''(t) - u''(t)v'(t)}{u(t)v'(t) - u'(t)v(t)} > 0.$$

For a unit vector $y = c(t) \in S$, we can take

$$y^\perp := \frac{1}{\sqrt{\sigma(t)}} \dot{c}(t).$$

The main scalar $\mathbf{I}(t) := \mathbf{I}(c(t))$ is given by

$$(39) \quad \mathbf{I}(t) = \frac{1}{\sqrt{\sigma(t)}} \frac{d}{dt} \left[\ln \frac{\sqrt{\sigma(t)}}{u(t)v'(t) - u'(t)v(t)} \right].$$

The above formulas also hold for singular Minkowski norms in dimension two.

We now consider a Randers norm $F = \alpha + \beta$ in V . Take an orthonormal basis $\{e_1, e_2\}$ for (V, α) such that $\beta(ue_1 + ve_2) = bu$, where $b = \|\beta\| := \sup_{\alpha(y)=1} \beta(y) < 1$. Then

$$(40) \quad F(ue_1 + ve_2) = \sqrt{u^2 + v^2} + bu.$$

The indicatrix $S = F^{-1}(1)$ is an ellipse determined by the following equation

$$(1 - b^2)^2 \left(u + \frac{b}{1 - b^2} \right)^2 + (1 - b^2)v^2 = 1.$$

Parameterize S by $c(t) = u(t)e_1 + v(t)e_2$

$$u(t) = -\frac{b}{1 - b^2} + \frac{1}{1 - b^2} \cos(t), \quad v(t) = \frac{1}{\sqrt{1 - b^2}} \sin(t).$$

Plugging $u(t)$ and $v(t)$ into (39), we obtain

$$(41) \quad \mathbf{I}(t) = -\frac{3}{2} \frac{b \sin(t)}{\sqrt{1 - b \cos(t)}}.$$

It is easy to see that

$$(42) \quad \|\mathbf{C}\| = \max_{F(y)=1} |\mathbf{I}(y)| = \max_{0 \leq t \leq 2\pi} |\mathbf{I}(t)| \leq \frac{3}{\sqrt{2}} \sqrt{1 - \sqrt{1 - b^2}}.$$

We obtain the same bound for $\mathbf{C}_y$ as in Exercise 11.2.6 in [BCS].

Now we consider a Randers norm $F = \alpha + \beta$ in an n -dimensional vector space V . We claim that the Cartan torsion still satisfies (36). To prove (36), we just need to simplify the problem to the two-dimensional case. Let y_0, v_0 with $F(y_0) = 1$ and $g_{y_0}(v_0, v_0) = 1$ such that

$$\|\mathbf{C}\| = \mathbf{C}_{y_0}(v_0, v_0, v_0).$$

Let $\bar{V} = \text{span}\{y_0, v_0\}$ and $\bar{F} := F|_{\bar{V}}$. The Cartan torsion $\bar{\mathbf{C}}$ of $\bar{F}$ satisfies

$$\mathbf{C}_{y_0}(v_0, v_0, v_0) = \frac{1}{4} \frac{\partial^3}{\partial s^3} [F^2(y_0 + sv_0)]|_{s=0} = \bar{\mathbf{C}}_{y_0}(v_0, v_0, v_0).$$

Let $\bar{\alpha} := \alpha|_{\bar{V}}$ and $\bar{\beta} = \beta|_{\bar{V}}$. We have

$$\|\bar{\beta}\| = \sup_{\bar{\alpha}(y)=1} \bar{\beta}(y) \leq \sup_{\alpha(y)=1} \beta(y) = \|\beta\|.$$

Let $\bar{\mathbf{I}}(y_0)$ denote the main scalar of $\bar{F}$ at y_0 . By the above argument, we have

$$(43) \quad \|\bar{\mathbf{C}}\| = \max_{\bar{F}(y)=1} |\bar{\mathbf{I}}(y)| \leq \frac{3}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \|\bar{\beta}\|^2}}.$$

Thus

$$\|\mathbf{C}\| \leq \|\bar{\mathbf{C}}\| \leq \frac{3}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \|\bar{\beta}\|^2}} \leq \frac{3}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \|\beta\|^2}}.$$

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