

Generalizations of the Cauchy determinant

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Abstract. By means of partial fraction expansions, several determinant evaluations are established. They may be considered as generalizations of the Cauchy determinant formula. Applications to the determinant identities of the matrices with trigonometric entries are presented.

0. Preliminaries

For two polynomials $P(y)$ and $Q(y)$, define a matrix of order $n \times n$ with variables $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$, and its determinant by

$$(0.1) \quad \Omega(P, Q) = \det \left[\frac{P(y_j) + x_i Q(y_j)}{1 - x_i y_j} \right]_{1 \leq i, j \leq n}.$$

When $P(y) = 1$ and $Q(y) = 0$, it reduces to the Cauchy determinant [2, §1.4]

$$(0.2) \quad \det \left[\frac{1}{1 - x_i y_j} \right]_{1 \leq i, j \leq n} = \frac{\Delta(x) \Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)}$$

where $\Delta(x)$ is the evaluation of the Vandermonde determinant

$$(0.3) \quad \Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \det \left[x_i^{n-j} \right]_{1 \leq i, j \leq n}.$$

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The difference between the i -th row and the first row of $\Omega(P, Q)$ reads as

$$\frac{(x_i - x_1)\{Q(y_j) + y_j P(y_j)\}}{(1 - x_1 y_j)(1 - x_i y_j)}, \quad i = 2, 3, \dots, n, \quad j = 1, 2, \dots, n.$$

Now Laplace expansion of the resulting determinant with respect to the first row may be stated as

$$(0.4a) \quad \Omega(P, Q) = \sum_{k=1}^n (-1)^{k+1} \frac{P(y_k) + x_1 Q(y_k)}{1 - x_1 y_k} \\ (0.4b) \quad \times \det \left[\frac{(x_i - x_1)\{Q(y_j) + y_j P(y_j)\}}{(1 - x_1 y_j)(1 - x_i y_j)} \right]_{i \neq 1, j \neq k}.$$

Extracting the common factors from rows and columns, we may rewrite (0.4b) as

$$\det \left[\frac{(x_i - x_1)\{Q(y_j) + y_j P(y_j)\}}{(1 - x_1 y_j)(1 - x_i y_j)} \right]_{i \neq 1, j \neq k} \\ = \prod_{j \neq k} \frac{Q(y_j) + y_j P(y_j)}{1 - x_1 y_j} \prod_{i=2}^n (x_i - x_1) \det \left[\frac{1}{1 - x_i y_j} \right]_{i \neq 1, j \neq k}.$$

where the Cauchy determinant may be used to derive

$$\det \left[\frac{1}{1 - x_i y_j} \right]_{i \neq 1, j \neq k} \\ = (-1)^{k-1} \frac{\prod_{i=2}^n (1 - x_i y_k)}{\prod_{j \neq k} (y_j - y_k)} \frac{\Delta(x) \Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \frac{\prod_{j=1}^n (1 - x_1 y_j)}{\prod_{i=2}^n (x_i - x_1)}.$$

Substituting these expressions into (0.4b), we find a finite summation formula for $\Omega(P, Q)$

Lemma.

$$(0.5a) \quad \Omega(P, Q) = \frac{\Delta(x) \Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \prod_{\ell=1}^n \{Q(y_\ell) + y_\ell P(y_\ell)\}$$

$$(0.5b) \quad \times \sum_{k=1}^n \frac{P(y_k) + x_1 Q(y_k)}{Q(y_k) + y_k P(y_k)} \frac{\prod_{i \neq 1} (1 - x_i y_k)}{\prod_{j \neq k} (y_j - y_k)}.$$

By evaluating (0.5b) with partial fraction method [1], we will establish the main theorem in the first section. Several examples will be demonstrated in the second and the third sections. In the last section, two determinant identities of the matrices with trigonometric entries are displayed as applications.

1. The main theorem and proof

Proposition. *For two polynomials $P(y)$ and $Q(y)$ with $\deg(Py+Q) > \deg(P)$, let*

$$\varepsilon(P, Q) = \begin{cases} 0, & \deg(P) \geq \deg(Q) \\ \frac{L_c(Q)}{L_c(Py+Q)}, & \deg(P) < \deg(Q) \end{cases}$$

where $L_c(Q)$ and $L_c(Py+Q)$ denote the leading coefficients of $Q(y)$ and $P(y)y + Q(y)$ respectively. Suppose that $P(y)y + Q(y)$ has m distinct roots $\{\beta_\lambda\}_{\lambda=1}^m$ such that

$$Q(y) + yP(y) = L_c(Py+Q) \prod_{\lambda=1}^m (y - \beta_\lambda).$$

We may express (0.5b) as

$$(1.1) \quad \sum_{k=1}^n \frac{P(y_k) + x_1 Q(y_k)}{Q(y_k) + y_k P(y_k)} \frac{\prod_{i \neq 1} (1 - x_i y_k)}{\prod_{j \neq k} (y_j - y_k)} = \varepsilon(P, Q) x_1 x_2 \dots x_n \\ + L_c^{-1}(Py+Q) \sum_{\ell=1}^m \frac{P(\beta_\ell)}{\prod_{j \neq \ell} (\beta_\ell - \beta_j)} \prod_{i=1}^n \frac{(1 - x_i \beta_\ell)}{(y_i - \beta_\ell)}.$$

Replacing (0.5b) by the formula in the proposition, we get

Theorem. *Assume the condition and notation from the proposition. We have determinant evaluation formula*

$$(1.2) \quad \Omega(P, Q) = \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \prod_{k=1}^n \{Q(y_k) + y_k P(y_k)\} \\ \times \left\{ \varepsilon(P, Q) \prod_{i=1}^n x_i + L_c^{-1}(Py+Q) \sum_{\ell=1}^m \frac{P(\beta_\ell)}{\prod_{j \neq \ell} (\beta_\ell - \beta_j)} \prod_{i=1}^n \frac{(1 - x_i \beta_\ell)}{(y_i - \beta_\ell)} \right\}.$$

PROOF. Under the condition of the proposition, we may make the expansion in partial fractions

$$\begin{aligned}
 (1.3) \quad f(y) &:= \frac{P(y) + x_1 Q(y)}{Q(y) + y P(y)} \frac{\prod_{i \neq 1} (1 - x_i y)}{\prod_{j \neq k} (y_j - y)} \\
 &= A + \sum_{\kappa \neq k} \frac{B_\kappa}{y_\kappa - y} + \sum_{\ell=1}^m \frac{C_\ell}{y - \beta_\ell}
 \end{aligned}$$

with the coefficients determined by

$$(1.4a) \quad A = \lim_{y \rightarrow \infty} f(y) = \varepsilon(P, Q) x_1 x_2 \dots x_n$$

$$\begin{aligned}
 (1.4b) \quad B_\kappa &= \lim_{y \rightarrow y_\kappa} (y_\kappa - y) f(y) \\
 &= \frac{P(y_\kappa) + x_1 Q(y_\kappa)}{Q(y_\kappa) + y_\kappa P(y_\kappa)} \frac{\prod_{i \neq 1} (1 - x_i y_\kappa)}{\prod_{j \neq \kappa, k} (y_j - y_\kappa)}
 \end{aligned}$$

$$\begin{aligned}
 (1.4c) \quad C_\ell &= \lim_{y \rightarrow \beta_\ell} (y - \beta_\ell) f(y) \\
 &= L_c^{-1}(Py + Q) \frac{P(\beta_\ell) + x_1 Q(\beta_\ell)}{\prod_{j \neq \ell} (\beta_\ell - \beta_j)} \frac{\prod_{i \neq 1}^n (1 - x_i \beta_\ell)}{\prod_{i \neq k}^n (y_i - \beta_\ell)}.
 \end{aligned}$$

By means of series-rearrangement and

$$P(\beta_\ell) + x_1 Q(\beta_\ell) = (1 - x_1 \beta_\ell) P(\beta_\ell), \quad \ell = 1, 2, \dots, m$$

we can deduce

$$\begin{aligned}
 \sum_{k=1}^n \sum_{\kappa \neq k} \frac{B_\kappa}{y_\kappa - y_k} &= - \sum_{\kappa=1}^n \sum_{k \neq \kappa} \frac{P(y_\kappa) + x_1 Q(y_\kappa)}{Q(y_\kappa) + y_\kappa P(y_\kappa)} \frac{\prod_{i \neq 1} (1 - x_i y_\kappa)}{\prod_{j \neq \kappa} (y_j - y_\kappa)} \\
 &= (1 - n) \sum_{\kappa=1}^n f(y_\kappa)
 \end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^n \sum_{\ell=1}^m \frac{C_\ell}{y_k - \beta_\ell} &= L_c^{-1}(Py + Q) \sum_{\ell=1}^m \sum_{k=1}^n \frac{P(\beta_\ell) + x_1 Q(\beta_\ell)}{(1 - x_1 y_\ell) \prod_{j \neq \ell} (\beta_\ell - \beta_j)} \\
&\quad \times \prod_{i=1}^n \frac{1 - x_i \beta_\ell}{y_i - \beta_\ell} \\
&= \frac{n}{L_c(Py + Q)} \sum_{\ell=1}^m \frac{P(\beta_\ell)}{\prod_{j \neq \ell} (\beta_\ell - \beta_j)} \prod_{i=1}^n \frac{1 - x_i \beta_\ell}{y_i - \beta_\ell}
\end{aligned}$$

which permit us to sum (1.3a–1.3b) for $y = y_k$, $k = 1, 2, \dots, n$. After some trivial modification, the resulting relation reads as

$$\begin{aligned}
(1.5) \quad \sum_{k=1}^n f(y_k) &= \varepsilon(P, Q) x_1 x_2 \dots x_n \\
&\quad + L_c^{-1}(Py + Q) \sum_{\ell=1}^m \frac{P(\beta_\ell)}{\prod_{j \neq \ell} (\beta_\ell - \beta_j)} \prod_{i=1}^n \frac{1 - x_i \beta_\ell}{y_i - \beta_\ell}.
\end{aligned}$$

It is a restatement of the formula in the proposition. This completes the proof of the theorem. $\square$

2. Determinant evaluations

It is clear that the Cauchy determinant (0.2) is a special case of the theorem for $P(y) = 1$ and $Q(y) = 0$. Now we will derive, by means of (1.2), several determinant evaluations. For the computations involved are quite routine, it is not necessary to show in detail.

Example 2.1 (Determinant evaluation).

$$\begin{aligned}
\text{For} \quad P(y) &= a, & Q(y) &= b + cy \\
\varepsilon(P, Q) &= c/(a + c), & L_c(Py + Q) &= a + c
\end{aligned}$$

there holds determinant identity

$$\begin{aligned}
\det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cx_i y_j}{1 - x_i y_j} \right] &= \frac{\Delta(x) \Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \prod_{k=1}^n \{b + (a + c)y_k\} \\
&\quad \times \left\{ \frac{c}{a + c} \prod_{\iota=1}^n x_\iota + \frac{a}{a + c} \prod_{\ell=1}^n \frac{a + c + bx_\ell}{b + (a + c)y_\ell} \right\}.
\end{aligned}$$

Example 2.2 (Determinant evaluation).

$$\begin{aligned} \text{For } P(y) &= a + cy, & Q(y) &= b \\ \varepsilon(P, Q) &= 0, & L_c(Py + Q) &= c \end{aligned}$$

there holds determinant identity

$$\begin{aligned} \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cy_j}{1 - x_i y_j} \right] &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \prod_{k=1}^n \{b + ay_k + cy_k^2\} \\ &\times \left\{ \frac{a + c\alpha}{c(\alpha - \beta)} \prod_{\ell=1}^n \frac{1 - x_\ell \alpha}{y_\ell - \alpha} - \frac{a + c\beta}{c(\alpha - \beta)} \prod_{\ell=1}^n \frac{1 - x_\ell \beta}{y_\ell - \beta} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = b + ay + cy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 2.3 (Determinant evaluation).

$$\begin{aligned} \text{For } P(y) &= a + cy, & Q(y) &= b + dy \\ \varepsilon(P, Q) &= 0, & L_c(Py + Q) &= c \end{aligned}$$

there holds determinant identity

$$\begin{aligned} \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cy_j + dx_i y_j}{1 - x_i y_j} \right] &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \prod_{k=1}^n \{b + (a + d)y_k + cy_k^2\} \\ &\times \left\{ \frac{a + c\alpha}{c(\alpha - \beta)} \prod_{\ell=1}^n \frac{1 - x_\ell \alpha}{y_\ell - \alpha} - \frac{a + c\beta}{c(\alpha - \beta)} \prod_{\ell=1}^n \frac{1 - x_\ell \beta}{y_\ell - \beta} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = b + (a + d)y + cy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 2.4 (Determinant evaluation).

$$\begin{aligned} \text{For } P(y) &= a, & Q(y) &= b + cy + dy^2 \\ \varepsilon(P, Q) &= 1, & L_c(Py + Q) &= d \end{aligned}$$

there holds determinant identity

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cx_i y_j + dx_i y_j^2}{1 - x_i y_j} \right] \\ &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \prod_{k=1}^n \{b + (a + c)y_k + dy_k^2\} \\ & \times \left\{ x_1 x_2 \dots x_n + \frac{a/d}{\alpha - \beta} \prod_{\ell=1}^n \frac{1 - x_\ell \alpha}{y_\ell - \alpha} - \frac{a/d}{\alpha - \beta} \prod_{\ell=1}^n \frac{1 - x_\ell \beta}{y_\ell - \beta} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = b + (a + c)y + dy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 2.5 (Determinant evaluation).

$$\begin{aligned} \text{For } P(y) &= a + cy, & Q(y) &= b + dy + ey^2 \\ \varepsilon(P, Q) &= e/(c + e), & L_c(Py + Q) &= c + e \end{aligned}$$

there holds determinant identity

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cy_j + dx_i y_j + ex_i y_j^2}{1 - x_i y_j} \right] \\ &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \prod_{k=1}^n \{b + (a + d)y_k + (c + e)y_k^2\} \\ & \times \frac{1}{c + e} \left\{ e x_1 x_2 \dots x_n + \frac{a + c\alpha}{\alpha - \beta} \prod_{\ell=1}^n \frac{1 - x_\ell \alpha}{y_\ell - \alpha} - \frac{a + c\beta}{\alpha - \beta} \prod_{\ell=1}^n \frac{1 - x_\ell \beta}{y_\ell - \beta} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = b + (a + d)y + (c + e)y^2$$

has two distinct zeros $\{\alpha, \beta\}$.

3. Reformulations

Replacing x_k by $-1/x_k$ for $k = 1, 2, \dots, n$, we may restate the theorem as

Corollary. *For two polynomials $P(y)$ and $Q(y)$, assume the condition and notation from the proposition. We have determinant evaluation formula*

$$(3.1) \quad \det_{1 \leq i, j \leq n} \left[\frac{Q(y_j) - x_i P(y_j)}{x_i + y_j} \right] = \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \prod_{k=1}^n \{P(y_k)y_k + Q(y_k)\} \\ \times \left\{ \varepsilon(P, Q) + L_c^{-1}(Py + Q) \sum_{\ell=1}^m \frac{P(\beta_\ell)}{\prod_{j \neq \ell} (\beta_\ell - \beta_j)} \prod_{i=1}^n \frac{\beta_\ell + x_i}{\beta_\ell - y_i} \right\}.$$

When $P(y) = 0$ and $Q(y) = 1$, the formula in the corollary reduces to another determinant identity of Cauchy

$$\det_{1 \leq i, j \leq n} \left[\frac{1}{x_i + y_j} \right] = \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)}.$$

Similar to the last section, the following determinant evaluations may be established consequently.

Example 3.1 (Determinant evaluation).

$$\text{For} \quad \begin{aligned} P(y) &= -b, & Q(y) &= a + cy \\ \varepsilon(P, Q) &= c/(c - b), & L_c(Py + Q) &= c - b \end{aligned}$$

there holds determinant identity

$$\det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cy_j}{x_i + y_j} \right] = \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \prod_{k=1}^n \{a - (b - c)y_k\} \\ \times \left\{ \frac{c}{c - b} + \frac{b}{b - c} \prod_{\ell=1}^n \frac{a + (b - c)x_\ell}{a - (b - c)y_\ell} \right\}.$$

Example 3.2 (Determinant evaluation).

$$\begin{aligned} \text{For} \quad P(y) &= -b - cy, & Q(y) &= a \\ \varepsilon(P, Q) &= 0, & L_c(Py + Q) &= -c \end{aligned}$$

there holds determinant identity

$$\begin{aligned} \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cx_i y_j}{x_i + y_j} \right] &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \prod_{k=1}^n \{a - by_k - cy_k^2\} \\ &\times \left\{ \frac{b + c\alpha}{c(\alpha - \beta)} \prod_{\ell=1}^n \frac{\alpha + x_\ell}{\alpha - y_\ell} - \frac{b + c\beta}{c(\alpha - \beta)} \prod_{\ell=1}^n \frac{\beta + x_\ell}{\beta - y_\ell} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = a - by - cy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 3.3 (Determinant evaluation).

$$\begin{aligned} \text{For} \quad P(y) &= -b - dy, & Q(y) &= a + cy \\ \varepsilon(P, Q) &= 0, & L_c(Py + Q) &= -d \end{aligned}$$

there holds determinant identity

$$\begin{aligned} \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cy_j + dx_i y_j}{x_i + y_j} \right] &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \prod_{k=1}^n \{a - (b - c)y_k - dy_k^2\} \\ &\times \left\{ \frac{b + d\alpha}{d(\alpha - \beta)} \prod_{\ell=1}^n \frac{\alpha + x_\ell}{\alpha - y_\ell} - \frac{b + d\beta}{d(\alpha - \beta)} \prod_{\ell=1}^n \frac{\beta + x_\ell}{\beta - y_\ell} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = a - (b - c)y - dy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 3.4 (Determinant evaluation).

$$\begin{aligned} \text{For} \quad P(y) &= -b, & Q(y) &= a + cy + dy^2 \\ \varepsilon(P, Q) &= 1, & L_c(Py + Q) &= d \end{aligned}$$

there holds determinant identity

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cy_j + dy_j^2}{x_i + y_j} \right] \\ &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \prod_{k=1}^n \{a - (b - c)y_k + dy_k^2\} \\ & \times \left\{ 1 - \frac{b/d}{\alpha - \beta} \prod_{\ell=1}^n \frac{\alpha + x_\ell}{\alpha - y_\ell} + \frac{b/d}{\alpha - \beta} \prod_{\ell=1}^n \frac{\beta + x_\ell}{\beta - y_\ell} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = a - (b - c)y + dy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 3.5 (Determinant evaluation).

$$\begin{aligned} \text{For} \quad P(y) &= -b - dy, & Q(y) &= a + cy + ey^2 \\ \varepsilon(P, Q) &= e/(e - d), & L_c(Py + Q) &= e - d \end{aligned}$$

there holds determinant identity

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\frac{a + bx_i + cy_j + dx_i y_j + ey_j^2}{x_i + y_j} \right] \\ &= \frac{\Delta(x)\Delta(y)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \prod_{k=1}^n \{a - (b - c)y_k - (d - e)y_k^2\} \\ & \times \left\{ \frac{e}{e - d} + \frac{b + d\alpha}{(d - e)(\alpha - \beta)} \prod_{\ell=1}^n \frac{\alpha + x_\ell}{\alpha - y_\ell} - \frac{b + d\beta}{(d - e)(\alpha - \beta)} \prod_{\ell=1}^n \frac{\beta + x_\ell}{\beta - y_\ell} \right\} \end{aligned}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = a - (b - c)y - (d - e)y^2$$

has two distinct zeros $\{\alpha, \beta\}$.

4. Applications

As applications, we will derive two interesting determinants of the matrices with trigonometric entries. Throughout this section, denote by λ the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ for n given real numbers $\{\lambda_k\}_{k=1}^n$.

In Example 2.1, let

$$\begin{aligned} a &= c = \sqrt{-1}, & b &= 0 \\ x_k &= -y_k = e^{2\lambda_k\theta\sqrt{-1}}. \end{aligned}$$

Then it is trivial to verify that

$$\begin{aligned} \det_{1 \leq i, j \leq n} \left[\frac{a + cx_i y_j}{1 - x_i y_j} \right] &= \det_{1 \leq i, j \leq n} \tan(\lambda_i + \lambda_j)\theta \\ \Delta(x) &= \prod_{1 \leq i < j \leq n} \sin(\lambda_j - \lambda_i)\theta \left\{ -2\sqrt{-1}e^{(\lambda_i + \lambda_j)\theta\sqrt{-1}} \right\} \\ \Delta(y) &= \prod_{1 \leq i < j \leq n} \sin(\lambda_j - \lambda_i)\theta \left\{ 2\sqrt{-1}e^{(\lambda_i + \lambda_j)\theta\sqrt{-1}} \right\} \\ \prod_{1 \leq i, j \leq n} (1 - x_i y_j) &= \prod_{1 \leq i, j \leq n} \cos(\lambda_i + \lambda_j)\theta \left\{ 2e^{(\lambda_i + \lambda_j)\theta\sqrt{-1}} \right\}. \end{aligned}$$

Substituting these relations into Example 2.1 and performing a little simplification, we obtain a general determinant identity

$$\begin{aligned} (4.1) \quad \det_{1 \leq i, j \leq n} [\tan(\lambda_i + \lambda_j)\theta] &= (-1)^{\lfloor n/2 \rfloor} \frac{\prod_{1 \leq i < j \leq n} \sin^2(\lambda_j - \lambda_i)\theta}{\prod_{1 \leq i, j \leq n} \cos(\lambda_i + \lambda_j)\theta} \\ &\quad \times \begin{cases} \cos(2\lambda\theta), & n \equiv 0 \pmod{2} \\ \sin(2\lambda\theta), & n \equiv 1 \pmod{2}. \end{cases} \end{aligned}$$

For $\lambda_k = k$, $k = 1, 2, \dots, n$, it reduces to a determinant evaluation

$$(4.2) \quad \det_{1 \leq i, j \leq n} [\tan(i+j)\theta] = (-1)^{\lfloor n/2 \rfloor} \prod_{k=1}^n \frac{\sin^{2n-2k} k\theta}{\cos^{k-1} k\theta \cos^{1+n-k}(n+k)\theta} \\ \times \begin{cases} \cos n(n+1)\theta, & n \equiv 0 \pmod{2} \\ \sin n(n+1)\theta, & n \equiv 1 \pmod{2}. \end{cases}$$

When $\lambda_k = k - 1/2$, $k = 1, 2, \dots, n$, we get another determinant evaluation

$$(4.3) \quad \det_{1 \leq i, j \leq n} [\tan(i+j-1)\theta] = (-1)^{\lfloor n/2 \rfloor} \prod_{k=1}^n \frac{\sin^{2n-2k} k\theta}{\cos^k k\theta \cos^{n-k}(n+k)\theta} \\ \times \begin{cases} \cos n^2\theta, & n \equiv 0 \pmod{2} \\ \sin n^2\theta, & n \equiv 1 \pmod{2} \end{cases}$$

which is due to STANLEY RABINOWITZ [3].

Instead, if we put in Example 2.1

$$a = c = -\sqrt{-1}, \quad b = 0$$

$$x_k = y_k = e^{2\lambda_k \theta \sqrt{-1}}$$

then it is not hard to check that

$$\det_{1 \leq i, j \leq n} \left[\frac{a + cx_i y_j}{1 - x_i y_j} \right] = \det_{1 \leq i, j \leq n} [\cot(\lambda_i + \lambda_j)\theta] \\ \Delta(x) = \Delta(y) = \prod_{1 \leq i < j \leq n} \sin(\lambda_j - \lambda_i)\theta \left\{ -2\sqrt{-1}e^{(\lambda_i + \lambda_j)\theta\sqrt{-1}} \right\} \\ \prod_{1 \leq i, j \leq n} (1 - x_i y_j) = \prod_{1 \leq i, j \leq n} \sin(\lambda_i + \lambda_j)\theta \left\{ -2\sqrt{-1}e^{(\lambda_i + \lambda_j)\theta\sqrt{-1}} \right\}.$$

Making these substitutions in Example 2.1, we get another general determinant identity

$$(4.4) \quad \det_{1 \leq i, j \leq n} [\cot(\lambda_i + \lambda_j)\theta] = \cos(2\lambda\theta) \frac{\prod_{1 \leq i < j \leq n} \sin^2(\lambda_j - \lambda_i)\theta}{\prod_{1 \leq i, j \leq n} \sin(\lambda_i + \lambda_j)\theta}.$$

For $\lambda_k = k$ and $k - 1/2$, $k = 1, 2, \dots, n$, we deduce from it two determinant evaluations

$$(4.5a) \quad \det_{1 \leq i, j \leq n} [\cot(i+j)\theta] = \cos n(n+1)\theta \prod_{k=1}^n \frac{\sin^{1+2n-3k} k\theta}{\sin^{1+n-k}(n+k)\theta}$$

$$(4.5b) \quad \det_{1 \leq i, j \leq n} [\cot(i+j-1)\theta] = \cos(n^2\theta) \prod_{k=1}^n \frac{\sin^{2n-3k} k\theta}{\sin^{n-k}(n+k)\theta}.$$

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