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On the exponential diophantine equation $(m^3 - 3m)^x + (3m^2 - 1)^y = (m^2 + 1)^z$

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Abstract. Let m be a positive integer. In this paper we prove that if $2 \parallel m$ and $m > 206$, then the equation $(m^3-3m)^x+(3m^2-1)^y = (m^2+1)^z$ has only one positive integer solution $(x, y, z) = (2, 2, 3)$ satisfying $x > 1$, $y > 1$ and $z > 1$.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. Let a, b, c, m, n, r be fixed positive integers satisfying

$$(1)\quad a^m + b^n = c^r, \quad \gcd(a,b) = 1,\ a > 1,\ b > 1,\ m > 1,\ n > 1,\ r > 1.$$

In 1994, Terai [4] conjectured that the equation

$$(2)\qquad a^x + b^y = c^z, \qquad x, y, z \in \mathbb{N},\ x > 1,\ y > 1,\ z > 1$$

has only one solution $(x, y, z) = (m, n, r)$. This is a rather difficult question. In [4], Terai considered the case that a, b and c can be expressed as

$$(3)\qquad a = m^3 - 3m, \quad b = 3m^2 - 1, \quad c = m^2 + 1,$$

where m is a positive integer with $2 \mid m$. Then (1) holds for $(m, n, r) = (2, 2, 3)$. Terai [4] showed that if b is a prime and there exists a prime p

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such that $p \mid m^2$ and $3 \mid e$, where e is the order of 2 modulo p, then (2) has only one solution $(x, y, z) = (2, 2, 3)$. Afterwards, the author [2] proved that if $2 \parallel m$ and b is a prime, then (2) has only one solution $(x, y, z) = (2, 2, 3)$. In this paper, we prove a general result as follows:

Theorem. *Let a, b, c be fixed positive integers satisfying* (3). *If $2 \parallel m$ and $m > 206$, then* (2) *has only solution $(x, y, z) = (2, 2, 3)$.*

The remaining cases $2 \leq m \leq 206$ in our theorem have been solved. A detailed proof will be given in an other paper.

2. Preliminaries

Lemma 1 ([3, pp. 12–13]). *Every solution (X, Y, Z) of the equation*

$$X^2 + Y^2 = Z^2 \qquad X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1,\ 2 \mid X \tag{4}$$

can be expressed as

$$X = 2uv, \quad Y = u^2 - v^2, \quad Z = u^2 + v^2,$$

where u, v, are positive integers satisfying

$$u > v, \qquad \gcd(u, v) = 1,\ 2 \mid uv. \tag{5}$$

Lemma 2 ([3, Theorem 4.2]). *The equation*

$$X^4 - Y^4 = Z^2, \qquad X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1$$

has no solution (X, Y, Z).

Lemma 3. *Let a, b, c be a positive integers satisfying* (3). *If $2 \parallel m$ and (x, y, z) is a solution of* (2) *with $(x, y, z) \neq (2, 2, 3)$, then we have either*

$$x = 2, \quad 2 \mid y, \quad y \geq 6, \quad 2 \nmid z \tag{6}$$

or

$$x = 4, \quad 2 \parallel y, \quad y \geq 10, \quad 2 \parallel z. \tag{7}$$

Proof. By the proof of [2, Theorem], we have $2 \mid x$ and $2 \mid y$. If $2 \nmid z$, then $b^y \equiv 1 \pmod 8$ and $c^z \equiv 5 \pmod 8$. It implies that $x = 2$.

Further, since $(x,y,z) \neq (2,2,3)$, we get $y \geq 4$ and $z > 3$. If $y = 4$, then we have $a^2 + b^4 \equiv 0 \pmod{c^z}$ and $a^2 + b^2 \equiv 0 \pmod{c^3}$. Hence, we get $b^2 \equiv 1 \pmod{c^3}$ and $b^2 - 1 \geq c^3 = a^2 + b^2 > 1 + b^2$, a contradiction. So we have $y \geq 6$ and (6) holds.

If $2 \mid z$, then $(X,Y,Z) = (a^{x/2}, b^{y/2}, c^{z/2})$ is a solution of (4). By Lemma 1, we get

$$a^{x/2} = 2uv, \quad b^{y/2} = u^2 - v^2, \quad c^{z/2} = u^2 + v^2, \tag{8}$$

where u, v are positive integers satisfying (5). Further, if $2 \mid y/2$, the from (8) we get $2 \nmid u$ and $4 \mid u$. It implies that $c^{z/2} \equiv 1 \pmod 8$ and $2 \mid z/2$. However, by Lemma 2, it is impossible. So we have $2 \parallel y$. Since $b \equiv 3 \pmod 8$, by (8), we get $2 \parallel u$ and $2 \nmid u$. Hence, we obtain $2 \parallel z$ and $x = 4$ by (8). Further, if $y = 2$, then we have $a^4 + b^2 \equiv 0 \pmod{c^z}$ and $a^2 + b^2 \equiv 0 \pmod{c^3}$. It implies that $a^2 \equiv 1 \pmod{c^3}$ and $a^2 - 1 \geq c^3 = a^2 + b^2 > a^2 + 1$, a contradiction. Similarly, if $y = 6$, then we get $b^2 + 1 \equiv 0 \pmod{c^3}$. It is impossible. So we have $y \geq 10$ and (7) holds. The lemma is proved. $\square$

Lemma 4. *Let a, b, c be a positive integers satisfying* (3). *If (x,y,z) is a solution of* (2) *satisfying* (6) *or* (7), *then it satisfies*

$$z + 3y - 9 \equiv \pmod{2m^2} \tag{9}$$

or

$$z + 3y \equiv 0 \pmod{2m^2}. \tag{10}$$

Proof. If the solution (x,y,z) satisfies (6), then from (2) and (3) we get

$$(z + 3y - 9) + \left(\binom{z}{2} - 9\binom{y}{2} + 6 \right) m^2 \equiv 0 \pmod{m^4}. \tag{11}$$

It implies that $z + 3y - 9 \equiv 0 \pmod{m^2}$. Since $y > 2$ and $z > 3$, we have

$$z + 3y - 9 = tm^2, \qquad t \in \mathbb{N}. \tag{12}$$

Substitute (12) into (11), we get $t + 7z - 21 \equiv 0 \pmod{m^2}$. Since $2 \nmid z$ and $2 \mid m$, we obtain $2 \mid t$ and (9) holds by (12).

Similarly, if (x,y,z) satisfy (7), then we have

$$(13)\qquad (z+3y)+\left(\binom{z}{2}-9\binom{y}{2}+6\right)m^2\equiv 0 \pmod{m^4},$$

whence we get $z+3y\equiv 0 \pmod{m^2}$ and

$$(14)\qquad z+3y=tm^2,\qquad t\in\mathbb{N}.$$

By (13) and (14), we obtain $t+6y+6\equiv 0 \pmod{m^2}$. It implies that $2\mid t$ and (10) holds by (14). The lemma is proved. □

Lemma 5. *Let a_1, a_2, b_1, b_2 be positive integers satisfying $\min(a_1,a_2)\ge 10^3$. Further let $\Lambda=b_1\log a_1-b_2\log a_2$. If $\Lambda\ne 0$, then*

$$(15)\qquad \log|\Lambda|>-17.61(\log a_1)(\log a_2)(1.7735+B)^2,$$

where

$$(16)\qquad B\ge\max\left(8.445, 0.2257+\log\left(\frac{b_1}{\log a_2}+\frac{b_2}{\log a_1}\right)\right).$$

Proof. For any real number ρ with $\rho>1$, by [1, Theorem 2], we have

$$(17)\qquad \log|\Lambda|\ge-\frac{16A_1A_2}{9\lambda^3}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^2$$

$$\times\left(1+\frac{3}{2}\lambda^3(A_1^{-1}+A_2^{-1})\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-1}+\sqrt[3]{2}\lambda^{3/2}\left(A_1A_2\left(B+\lambda+\frac{\lambda^2}{4B}\right)\right)^{-1/2}\right.$$

$$\left.+\frac{9\lambda^3}{8A_1A_2}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-1}+\frac{9\lambda^3}{16A_1A_2}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-2}\log\left(A_1A_2(B+\lambda)^2/\lambda^2\right)\right)$$

$$+\frac{\lambda}{2}+\log\lambda-0.15,$$

where $\lambda=\log\rho$, $a_j\ge\max(2,2\lambda,(\rho+1)\log a_j)$ $(j=1,2)$,

$$B\ge\max\left(5\lambda, 1.56+\log\lambda+\log\left(\frac{b_1}{A_2}+\frac{b_2}{A_1}\right)\right).$$

We now choose $\rho=e^{1.689}$. Then $A_j\ge(e^{1.689}+1)\log a_j$ $(j=1,2)$ and B satisfies (16). Therefore, if $\min(a_1,a_2)\ge 10^3$, then from (17) we get (15). The lemma is proved. □

3. Proof of Theorem

We now assume that $2 \parallel m$ and $m > 206$. Then we have $m \geq 210$ and $b > c > 10^4$. Let (x, y, z) be a solution of (2) with $(x, y, z) \neq (2, 2, 3)$. By Lemma 3, it satisfies either (6) or (7).

If (6) holds, then we have

$$a^2 + b^y = c^z, \quad 2 \mid y, \quad y \geq 6, \quad 2 \nmid z, \quad z > 3. \tag{18}$$

By (18), we get

$$z \log c - y \log b = \frac{2a^2}{c^z + b^y} \sum_{i=0}^{\infty} \frac{1}{2i+1} \left(\frac{a^2}{c^z + b^y} \right)^{2i} < \frac{2a^2}{c^z}. \tag{19}$$

Let $\Lambda = z \log c - y \log b$. We see from (19) that

$$\log 2a^2 - \log |\Lambda| > z \log c. \tag{20}$$

Since $b > c > 10^4$, by Lemma 5, we have

$$\log |\Lambda| > -17.61 (\log b)(\log c)(1.7735 + B)^2, \tag{21}$$

where

$$B = \max \left(8.445, 0.2257 + \log \left(\frac{z}{\log b} + \frac{y}{\log c} \right) \right). \tag{22}$$

Further, if $8.445 \geq 0.2257 + \log(z/\log b + y/\log c)$, then we have

$$\frac{2z}{\log b} - \frac{2a^2}{c^z} < \frac{z}{\log b} + \frac{y}{\log c} \leq e^{8.2193} < 3712 - \frac{2a^2}{c^z}. \tag{23}$$

On the other hand, since $z \geq y + 1$ by (18), we get

$$4z - 12 \geq z + 3y - 9 \geq 2m^2, \tag{24}$$

by (9). The combination of (23) and (24) yields

$$m^2 + 6 < 3712 \log b < 3712 (\log 3 + \log m^2), \tag{25}$$

whence we obtain $m < 210$, a contradiction. So we have

$$8.445 < 0.2257 + \log \left(\frac{z}{\log b} + \frac{y}{\log c} \right). \tag{26}$$

Substitute (26) into (21) and (22), we get

$$\begin{aligned}
&1+17.61\left(1.9992+\log\frac{2z}{\log b}\right)^2\\
(27)\qquad &> \frac{\log(2a^2)}{(\log b)(\log c)}+17.61\left(1.9992+\log\left(\frac{z}{\log b}+\frac{y}{\log c}\right)\right)^2\\
&> \frac{z}{\log b},
\end{aligned}$$

whence we calculate that $m < 210$, a contradiction. Thus, if $2 \parallel m$ and $m > 206$, then (2) has no solution (x, y, z) satisfying (6).

Using the same method, we can prove that if $2 \parallel m$ and $m > 206$, then (2) has no solution (x, y, z) satisfying (7). Therefore, by Lemma 3, the theorem is proved.

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