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S-unit equations, linear recurrences and digit expansions

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. We consider diophantine equations the solutions of which have bounded digits. More precisely we deal with equations in algebraic number fields, canonical number systems and digital representations based on linear recurrences. Our main result is an extension of a recent result of H.-P. SCHLICKEWEI giving a bound for the number of solutions of such equations.

1. Introduction

In 1973 SENGE and STRAUS [SS] proved that the number of integers the sum of whose digits in each of the bases a and b lies below a fixed bound is finite if and only if a and b are multiplicatively independent. A quantitative version of this result was given by STEWART [St] using BAKER's method. In 1989 SCHLICKEWEI [Sch1] established the following extension of the problem to equations $\pm n_1 \pm n_2 \pm \ldots \pm n_k = 0$.

Theorem 0. *Let $k \geq 2$ and suppose that $b_1, \ldots, b_k$ are integers larger than 1 such that b_j and b_l are multipicatively independent for $j \neq l$. Let c be a positive integer and let $n_1, n_2, \ldots, n_k$ be non-negative integers such that for each j with $1 \leq j \leq k$ the number n_j has sum of digits in base b_j bounded by c. Then the equation*

$$\pm n_1 \pm n_2 \pm \ldots \pm n_k = 0$$

has only finitely many solutions. Moreover the number of solutions is bounded by

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$$\big(8(\omega+1)\big)^{2^{26c}(\omega+1)^6},$$

where ω denotes the number of prime factors of $b_1 \cdots b_k$.

The proof of this theorem uses the quantitative version of the theorem on S-unit equations, which is an application of SCHLICKEWEI's p-adic version of SCHMIDT's subspace theorem. For a detailed survey on S-unit equations see [EGST].

In the theory of automatic sequences digit expansions with respect to linear recurring sequences play an important role. In the papers [PT], [GT1] and [GT2] such expansions were studied in some detail. The main results are concerned to the asymptotic behaviour of the sum of digits function. In the present paper we extend the above theorem to such digital systems. Let $G = (G_k)$, $k = 0, 1, \ldots$ be a linear recurring sequence of order $d \geq 1$

$$G_{k+d} = a_1 G_{k+d-1} + a_2 G_{k+d-2} + \cdots + a_d G_k$$

with integral coefficients and integral initial values. For $d = 1$ we assume $G_0 = 1$ and $a_1 > 1$. For $d \geq 2$ we assume that the coefficients $a_1 \geq a_2 \geq \cdots \geq a_d > 0$ are non-increasing and

$$1 = G_0, \quad G_k \geq a_1 G_{k-1} + \cdots + a_k G_0 + 1 \quad \text{for} \quad k = 1, \ldots, d-1.$$

For an arbitrary positive integer n define $L = L(n)$ by $G_L \leq n < G_{L+1}$ and put $L(0) = 0$. Set $m_L = n$ and ([t] denoting the integral part)

$$\varepsilon_j = \left[\frac{m_j}{G_j}\right] \quad \text{and} \quad m_{j-1} = m_j - G_j \varepsilon_j \quad (1 \leq j \leq L) \quad \text{and} \quad \varepsilon_0 = m_0.$$

Hence we obtain a well-defined representation of any positive integer n in the form

$$n = \sum_{j=0}^{L(n)} \varepsilon_j G_j,$$

the so-called G-ary expansion of n with digits $\varepsilon_j = \varepsilon_j(n)$. Furthermore we define

$$s(n) = s_G(n) = \sum_{j=0}^{L(n)} \varepsilon_j,$$

the sum of G-ary digits of n.

Remark 1. The $(t+1)$-tuple $(\varepsilon_0, \ldots, \varepsilon_t)$ of non-negative integers is the sequence of G-ary digits of an integer if and only if

$$\sum_{j=0}^{k} \varepsilon_j G_j < G_{k+1} \quad \text{for all} \quad 0 \leq k < d-1$$

and

$$(\varepsilon_k, \dots, \varepsilon_{k-d+1}) < (a_1, \dots, a_d) \quad \text{for all} \quad 0 \le k \le t.$$

This characterization is proved in [PT] and [Fr]. It should be noted here that a much more general situation is treated in [GT2].

Remark 2. From a theorem of A. BRAUER [Br] it follows that the characteristic roots of linear recurrences with non-increasing coefficients are PV-numbers, i.e. algebraic integers such that all but one conjugates are in absolute value less than 1. Furthermore BRAUER has shown that the characteristic polynomial is irreducible in that case. In the sequel the in absolute value largest root of such a polynomial is called the dominating root.

In this paper we extend SCHLICKEWEI's theorem in two directions. The first is the above described G-ary representation of integers and the second is the radix representation of algebraic integers in certain number fields.

We introduce now the second concept based on KOVÁCS and PETHŐ [KP]. Let K be an algebraic number field and let Z_K denote its ring of integers. Assume that Z_K has a power integral basis. Then there exists an $\alpha \in Z_K$ such that every $\gamma \in Z_K - \{0\}$ can be written uniquely in the form

$$\gamma = \sum_{i=0}^{L(\gamma)} a_i \alpha^i,$$

where $a_i \in N_\alpha = \{0, 1, \dots, |N_{K/Q}(\alpha)| - 1\}$, $i = 0, \dots, L(\gamma)$; (see [KP]). The pair (α, N_α) is called a canonical number system or CNS in Z_K. We define the sum of digit function with respect to the CNS (α, N_α) as

$$s_{K,\alpha}(\gamma) = \sum_{i=0}^{L(\gamma)} a_i$$

for $\gamma \in Z_K - \{0\}$ and $s_K(0) = 0$.

Our main tool for extending SCHLICKEWEI's theorem is the following general result on S-unit equations.

Theorem 1. *Let K be an algebraic number field of degree d over the rationals Q. Let $\alpha_1, \dots, \alpha_k \in K$ be multiplicatively independent of degrees $m_1, \dots, m_k$ respectively, $\beta_{i,j} \in K - \{0\}$ for $i = 1, \dots, k, j = 1, \dots, m_i$ and set $M = m_1 + \dots + m_k$. Let S be a set of places of K (including the archimedian places) having s elements and such that $\|\alpha_i\|_v = 1$ for all $i = 1, \dots, k$ and $v \notin S$. Then the diophantine equation*

$$\sum_{\kappa=1}^{k} \sum_{i=1}^{m_\kappa} \beta_{\kappa,i} \alpha_\kappa^{\lambda_{\kappa,i}} = 0 \tag{1.1}$$

has at most

$$B(M) := (4sd!)^{2^{36Md!}s^6} \tag{1.2}$$

solutions in $(\lambda_{1,1}, \ldots, \lambda_{1,m_1}, \ldots, \lambda_{k,1}, \ldots, \lambda_{k,m_k}) \in Z^M$ *such that*

$$\sum_{i \in J} \beta_{\kappa,i} \alpha_\kappa^{\lambda_{\kappa,i}} \neq 0 \tag{1.3}$$

holds for every $J \subseteq \{1, \ldots, m_\kappa\}, \kappa = 1, \ldots, k$.

In section 2 we will establish some auxiliary results for PV-numbers and for the proof of Theorem 1. In section 3 we will complete the proof and we will show the following consequences of Theorem 1 which is a generalization of the above Theorem 0.

Theorem 2. *Let* $k \geq 2$ *and suppose that* $G^1, \ldots, G^k$ *are non-degenerate linear recurrences of order* $n_1, \ldots, n_k$*, respectively with non-increasing coefficients as considered above. Assume further that the dominating characteristic roots are multiplicatively independent. Let* $c \geq 0$ *and* $u_1, \ldots, u_k$ *be non-negative integers such that* $s_{G^j}(u_j) \leq c$ *for each* $j \leq k$*. Then the equation*

$$\pm u_1 \pm u_2 \pm \ldots \pm u_k = 0 \tag{1.4}$$

has only D *solutions where* D *depends only on the degree of the splitting field* $\mathcal{L}$ *of the product* $p(x)$ *of the characteristic polynomials of* $G^1, \ldots, G^k$ *and on the number of distinct prime ideal factors in* $\mathcal{L}$ *of the constant term of* $p(x)$.

Theorem 3. *Let* K *be an algebraic number field of degree* d *such that* Z_K *has a power integral basis. Let* $k \geq 2$ *and* $\alpha_1, \ldots, \alpha_k \in Z_K$ *be multiplicatively independent and such that* $(\alpha_i, N_{\alpha_i}), i = 1, \ldots, k$ *is a CNS in* Z_K*. Let* S *be the minimal set of places (including the archimedean ones) of* K *for which*

$$\prod_{v \in S} \|\alpha_i\|_v = 1$$

holds for $i = 1, \ldots, k$*. Let* s *denote the number of elements of* S *and let* c *be a positive integer and* $w_1, \ldots, w_k \in Z_K$ *such that* $s_{K,\alpha_j}(w_j) \leq c$ *holds for all* $1 \leq j \leq k$*. Then the equation*

$$w_1 + \ldots + w_k = 0$$

has only finitely many solutions.

2. Auxiliary Results

Let K be an algebraic number field of degree d over the rationals Q. Let S be a finite set of places of K containing all archimedean places and let s denote the cardinality of S. The element $\alpha \in K$ is called an S-unit if

$$\prod_{v \in S} \|\alpha\|_v = 1.$$

In the proof of Theorem 1 we use the following result of Schickewei [Sch1, Theorem 1.1].

Lemma 1. *Let $\alpha_1, \ldots, \alpha_m \in K - \{0\}$. Then the equation*

$$\alpha_1 y_1 + \ldots + \alpha_m y_m = 1$$

has at most

$$\gamma(m) := (4sd!)^{2^{36md!}s^6}$$

solutions in S-units $y_1, \ldots, y_m$ such that no proper subsum $\alpha_{i_1} y_{i_1} + \ldots + \alpha_{i_r} y_{i_r}$ vanishes.

In the proof of Theorem 2 we need the following two properties of PV-numbers. If α is an algebraic number of degree d then $\alpha = \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)}$ denote its conjugates. For a PV-number α we assume $|\alpha^{(1)}| > 1 > |\alpha^{(2)}| \geq \ldots \geq |\alpha^{(d)}|$.

Lemma 2. *Let α be a PV-number of degree d. If $d > 2$ then the conjugates of α are pairwise multiplicatively independent. If $d = 2$ and $\alpha^{(1)}$ and $\alpha^{(2)}$ are multiplicatively dependent then $|\alpha^{(1)}\alpha^{(2)}| = 1$.*

Proof. We consider first the case $d > 2$. Assume that there exist $1 \leq i < j \leq d$ and $u, v \in Z$ with $|u| + |v| \neq 0$ such that

$$\alpha^{(i)u} = \alpha^{(j)v}. \tag{2.1}$$

There exists an element σ in the Galois group G of the normal closure over Q of $Q(\alpha)$ with $\sigma(\alpha^{(i)}) = \alpha$. Applying this automorphism to (2.1) we see that we may assume without loss of generality $i = 1$ and $j > 1$.As α is not a root of unity we may assume $v < 0 < u$.

As $d > 2$ there exists a $\tau \in G$ with $\tau(\alpha^{(j)}) = \alpha^{(k)} \neq \alpha$. Applying τ to (2.1) we get

$$\alpha^{(l)u} = \alpha^{(k)v}.$$

We have $\alpha^{(l)} \neq \alpha$ since otherwise $\alpha^{(j)}/\alpha^{(k)}$ would be a root of unity, which is impossible. Hence $1 < |\alpha\alpha^{(l)}| < |\alpha|$ and $|\alpha^{(j)}\alpha^{(k)}| < |\alpha^{(j)}| < 1$. Multiplying (2.1) with the last equation we get

$$|\alpha|^u > |\alpha\alpha^{(l)}|^u = |\alpha^{(j)}\alpha^{(k)}|^v > |\alpha^{(j)}|^v,$$

in contradiction with (2.1). Hence the first assertion is true.

Let now $d = 2$. Applying the non-trivial automorphism of $Q(\alpha)$ to (2.1) we get

$$\alpha^{(2)u} = \alpha^v.$$

This equation together with (2.1) implies

$$|\alpha\alpha^{(2)}|^{u-v} = 1,$$

thus $|\alpha\alpha^{(2)}| = 1$ as asserted, and the lemma is proved.

Lemma 3. *Let α_1 and α_2 be multiplicatively independent PV-numbers of degree d_1 and d_2, respectively. Then $\alpha_1^{(i)}$ and $\alpha_2^{(j)}$ are multiplicatively independent for all pairs (i, j) with $1 \le i \le d_1$ and $1 \le j \le d_2$.*

Proof. This is similar to the proof of Lemma 2 except that we have to argue with the elements of the Galois group of the normal closure over Q of $Q(\alpha_1, \alpha_2)$.

3. Proof of Theorems

Theorem 1 is a generalization of the Proposition of [Sch1]. It would be possible to prove it by the generalization of the argument used by Schlickewei. But using Lemma 1 and the adapted version of the proof of Theorem 1 of Becker [Be] we have chosen a more direct way.

Proof of Theorem 1. The assertion is true for $k = 1$ and every $M \ge 1$ because then (1.1) has no solution satisfying (1.3).

Let $k = 2$ and $M = 2$, then $m_1 = m_2 = 1$ and (1.1) becomes

$$\beta_{1,1}\alpha_1^{\lambda_{1,1}} + \beta_{2,1}\alpha_2^{\lambda_{2,1}} = 0.$$

Thus

$$\frac{\alpha_1^{\lambda_{1,1}}}{\alpha_2^{\lambda_{2,1}}} = \frac{-\beta_{2,1}}{\beta_{1,1}}.$$

As α_1 and α_2 are multiplicatively independent, this equation has at most one solution in $(\lambda_{1,1}, \lambda_{2,1}) \in Z^2$. Since $B(2) > 1$ the assertion is true in this case too.

Assume now that $k \ge 2$ and Theorem 1 is true for any $2 \le M' < M$. Let $m_1 + \ldots + m_k = M$.

Put

$$m'_\kappa = \begin{cases} m_\kappa, & \text{if } \kappa < k \\ m_\kappa - 1, & \text{if } \kappa = k, \end{cases}$$

$$\hat{\beta}_{\kappa,i} = -\frac{\beta_{\kappa,i}}{\beta_{k,m_k}},$$

and

$$y_{\kappa,i} = \frac{\alpha_\kappa^{\lambda_{\kappa,i}}}{\alpha_k^{\lambda_{k,m_k}}}$$

for $\kappa = 1 \dots, k$; $i = 1, \dots, m_\kappa$.

The numbers $\alpha_1 \dots, \alpha_k$ are multiplicatively independent, hence the mapping $\varphi : Z^M \mapsto K^{M-1}$ defined by

$$(\lambda_{1,1}, \dots, \lambda_{1,m_1}, \dots, \lambda_{k,1}, \dots, \lambda_{k,m_k}) \mapsto (y_{1,1}, \dots, y_{1,m_1'}, \dots, y_{k,1}, \dots, y_{k,m_k'})$$

is injective.

Dividing (1.1) by $\beta_{k,m_k}\alpha_k^{\lambda_k m_k}$ we get

$$\sum_{\kappa=1}^{k}\sum_{i=1}^{m_\kappa'} \hat{\beta}_{\kappa,i} y_{\kappa,i} = 1. \tag{3.1}$$

The number of non-degenerate solutions (i.e. without a vanishing subsum) of (3.1) in S-units $y_{1,1}, \dots, y_{k,m_k'}$, is at most $\gamma(M-1)$. Hence the number of non-degenerate solutions of (1.1) is also at most $\gamma(M-1)$. Corresponding to the degenerate solutions of (1.1) there exist degenerate solutions of (3.1) satisfying the following system of equations

$$\sum_{\kappa=1}^{k}\sum_{i\in J_\kappa} \hat{\beta}_{\kappa,i} y_{\kappa,i} = 0 \tag{3.2'}$$

$$\sum_{\kappa=1}^{k}\sum_{i\in J_\kappa'} \hat{\beta}_{\kappa,i} y_{\kappa,i} = 1, \tag{3.2"}$$

where $J_\kappa \subseteq \{1, \dots, m_\kappa'\}$ and J_κ' denotes the complementary of J_κ with respect to $\{1, \dots, m_\kappa'\}$; $\kappa = 1, \dots, k$, furthermore

$$1 \le |J_1 \cup \dots \cup J_k| \le m_1' + \dots + m_k' = M - 1.$$

If $(y_{1,1}, \dots, y_{k,m_k'})$ is such an S-unit solution of (3.1) which belongs to the image of φ, then we can rewrite (3.2) as

$$\sum_{\kappa=1}^{k}\sum_{i\in J_\kappa} \beta_{\kappa,i}\alpha_\kappa^{\lambda_{\kappa,i}} = 0 \tag{3.3'}$$

$$\sum_{\kappa=1}^{k}\sum_{i\in J_\kappa'} \beta_{\kappa,i}\alpha_\kappa^{\lambda_{\kappa,i}} + \beta_{k,m_k}\alpha_k^{\lambda_{k,m_k}} = 0. \tag{3.3"}$$

The last equations have the same structure as (1.1), but both have at most $M-1$ summands. By the induction hypothesis, (3.3') has at most $B(|J_1|+\ldots+|J_k|) \le B(M-1)$ solutions and (3.3") has at most $B\left(M-(|J_1|+\ldots+|J_k|+1)\right) \le B(M-1)$ solutions which satisfy (1.3). For the choice of the sets $J_1,\ldots,J_k$ we have at most 2^{M-1} possibilities, hence

$$B(M) \le \gamma(M-1) + 2^{M-1}B(M-1)^2.$$

From this recursive inequality (1.2) follows immediately and Theorem 1 is proved.

Proof of Theorem 2. Let G^i satisfy the recurrence

$$G^i_{\lambda+n_i} = a_{i,1}G^i_{\lambda+n_i-1} + \ldots + a_{i,n_i}G^i_\lambda$$

with $i=1,\ldots,k$. We may assume after possible reordering that $G^1,\ldots,G^\ell$ with $\ell \le k$ are those with $n_i=2$ and $|a_{i,n_i}|=1$, $i=1,\ldots,\ell$. Furthermore, if $\ell < i \le k$, then either $n_i > 2$ or $n_i = 2$ and $|a_{i,n_i}| > 1$.

Let $\alpha_{i,1}$ denote the dominating zero of the characteristic polynomial of G^i with $1 \le i \le k$, and $\alpha_{i,j}$ denote the conjugates of $\alpha_{i,1}$, $1 \le i \le k, 1 \le j \le n_i$. There exist algebraic integers $\beta_{i,j}, 1 \le i \le k; 1 \le j \le n_i$ such that

$$G^i_\lambda = \sum_{j=1}^{n_i} \beta_{i,j}\alpha^\lambda_{i,j} \tag{3.4}$$

holds for any $\lambda \in Z_0^+$, where Z_0^+ denotes the set of non-negative integers. As $G^i_\lambda \in Z$ for all $\lambda \in Z_0^+$, $i=1,\ldots,k$ we have $\beta_{i,j} \in Q(\alpha_{i,j})$ and $\beta_{i,j}$ is a conjugate of $\beta_{i,1}$.

Let $u_1,\ldots,u_k \in Z_0^+$ satisfying (1.4). We are asking for solutions u_κ of (1.4) with sum of G^κ-ary digits bounded by c. Hence there exist positive integers $d_{\kappa,1},\ldots,d_{\kappa,m_\kappa}$ $(m_\kappa \le c)$ with $0 \le d_{\kappa,1}+\ldots+d_{\kappa,m_\kappa} \le c$ such that

$$u_\kappa = d_{\kappa,1}G^\kappa_{\lambda_{\kappa,1}} + \ldots + d_{\kappa,m_\kappa}G^\kappa_{\lambda_{\kappa,m_\kappa}}$$

with suitable non-negative integers $\lambda_{\kappa,i}$. Now we divide the set of solutions of (1.4) into classes by grouping together solutions u_κ corresponding to a fixed m_κ-tuple $d_{\kappa,1},\ldots,d_{\kappa,m_\kappa}$ with $1 \le \kappa \le k$.

Those solutions of (1.4) belonging to the same class are solutions of a fixed equation

$$\sum_{\kappa=1}^{k}(\pm 1)\sum_{i=1}^{m_\kappa} d_{\kappa,i}\sum_{j=1}^{n_\kappa}\beta_{\kappa,j}\alpha_{\kappa,j}^{\lambda_{\kappa,i}} = \sum_{\kappa=1}^{k}(\pm 1)\sum_{i=1}^{m_\kappa}\sum_{j=1}^{n_\kappa}(d_{\kappa,i}\beta_{\kappa,j})\alpha_{\kappa,j}^{\lambda_{\kappa,i}} = 0.$$

We split now the left hand side of the last equation into two summands and get

$$\sum_{\kappa=1}^{\ell}(\pm 1)\sum_{h=1}^{m_\kappa} d_{\kappa,h}(\beta_{\kappa,1}\alpha_{\kappa,1}^{\lambda_{\kappa,h}} + \beta_{\kappa,2}\alpha_{\kappa,2}^{\lambda_{\kappa,h}}) + \sum_{\kappa=\ell+1}^{k}(\pm 1)\sum_{j=1}^{n_\kappa}\sum_{h=1}^{m_\kappa} d_{\kappa,h}\beta_{\kappa,j}\alpha_{\kappa,j}^{\lambda_{\kappa,h}} = 0. \tag{3.5}$$

By lemmas 2 und 3 the algebraic numbers

$$\alpha_{1,1},\dots,\alpha_{\ell,1},\alpha_{\ell+1,1},\dots,\alpha_{\ell+1,n_{\ell+1}},\dots,\alpha_{k,1},\dots,\alpha_{k,n_k}$$

are pairwise multiplicatively independent and all these numbers belong to the field G defined in the theorem.

Let $p(x)$ be the product of the characteristic polynomials of $G^1,\dots,G^k$. Then the free term of $p(x)$ is $A=\prod_{i=1}^{k} a_{i,n_i}$. Let S be the minimal set of places of G such that $\prod_{v\in S}\|A\|_v = 1$. If g is the degree of G over the rationals then $|S| \le g(\omega(A)+1)$, where $\omega(A)$ denotes the number of distinct prime divisors of A. Further $\alpha_{\kappa,j}$ divides A in Z_G for all $1\le\kappa\le k$ and $1\le j\le n_j$. Thus $\prod_{v\in S}\|\alpha_{k,j}\| = 1$, i.e. $\alpha_{k,j}$ are S-units.

By Theorem 1 equation (3.5) has at most $B(m_1+\ldots+m_\ell+\sum_{\kappa=\ell+1}^{k} m_\kappa n_\kappa) \le B(c\sum_{\kappa=1}^{k} n_\kappa)$ solutions in

$$\lambda_{1,1},\lambda_{2,1},\dots,\lambda_{\ell,1},\lambda_{\ell+1,1},\dots,\lambda_{\ell+1,n_{\ell+1}},\dots,\lambda_{k,m_k}\in Z$$

such that

$$\sum_{h\in J_\kappa} d_{\kappa,h}\beta_{\kappa,j}\alpha_{\kappa,j}^{\lambda_{\kappa,h}} \neq 0 \tag{3.6}$$

holds for all $J_\kappa\subseteq\{1,\dots,m_\kappa\}, \kappa=\ell+1,\dots,k$ and

$$\sum_{h\in J_\kappa} d_{\kappa,h}\beta_{\kappa,1}^{\lambda_{\kappa,h}}\alpha_{\kappa,1}^{\lambda_{\kappa,h}} + \sum_{h\in J'_\kappa} d_{\kappa,h}\beta_{\kappa,2}\alpha_{\kappa,2}^{\lambda_{\kappa,h}} \neq 0 \tag{3.7}$$

holds for all $J_\kappa, J'_\kappa\subseteq\{1,\dots,m_\kappa\}, \kappa=1,\dots,\ell$.

Assume that there exists an $\ell<\kappa\le k, 1\le j\le n_\kappa$ and $J_\kappa\subseteq\{1,\dots,m_\kappa\}$ such that (3.6) does not hold, i.e.

$$\sum_{h\in J_\kappa} d_{\kappa,h}\beta_{\kappa,j}\alpha_{\kappa,j}^{\lambda_{\kappa,h}} = 0. \tag{3.8}$$

Taking conjugates in (3.8) and using that $d_{\kappa,h}$ are rational and $\beta_{\kappa,j}\in Q(\alpha_{\kappa,j})$ and $\beta_{\kappa,j}$ is a conjugate of $\beta_{\kappa,1}$, we conclude that (3.8) holds for

all $j = 1, \ldots, n_\kappa$. Thus

$$\sum_{h \in J_\kappa} d_{\kappa,h} G^\kappa_{\lambda_{\kappa,h}} = 0.$$

But this equation cannot hold because $d_{\kappa,h}$ and $G^\kappa_{\lambda_{\kappa,h}}$ are all positive. Hence (3.6) always is true.

Similarly one can see that (3.7) always is true and the proof of Theorem 2 is complete.

PROOF of Theorem 3. This is analogous, even more easy than the proof of Theorem 2. Hence it is left to the reader.

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