

Strong convergence theorems for $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$

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Abstract. Multiplier operators on the Hardy space $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ are investigated and Bernstein's inequality for multi-parameter trigonometric polynomials is verified. We prove that certain means of the partial sums of the multi-parameter trigonometric Fourier series are uniformly bounded operators from $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ to L_p ($1/2 < p \leq 1$). As a consequence we obtain strong convergence theorems concerning the partial sums. The dual inequalities are also verified and a Marcinkiewicz–Zygmund type inequalities is obtained for the $BMO(\mathbb{T} \times \cdots \times \mathbb{T})$ spaces.

1. Introduction

We introduce the d -dimensional Hardy space $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ by the $L_p(\mathbb{T}^d)$ norm of the non-tangential maximal function of a distribution on $\mathbb{T}^d$. It is known that the trigonometric system is not a basis in $L_1(\mathbb{T})$. Moreover, there exist functions in $H_1(\mathbb{T})$, the partial sums of which are not bounded in $L_1(\mathbb{T})$. SMITH [10] and recently BELINSKII [1] proved the following strong convergence result for one-parameter trigonometric Fourier series:

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\|s_k f - f\|_1}{k} = 0$$

where $f \in H_1(\mathbb{T})$ and $s_k f$ denotes the k -th partial sum of the Fourier series. This result for one-parameter Walsh–Fourier series can be found in SIMON [9].

Mathematics Subject Classification: Primary: 43A75, 42B05, 42B08; Secondary: 42B30.
Key words and phrases: Hardy spaces, multiplier operators, Bernstein's inequality, strong means.

This research was supported by OTKA No. F019633, FKFP No. 0228/1999 and by the Széchenyi Professorship.

Recently the author [12] generalized this result for two-parameter trigonometric Fourier series by taking the sum over a cone. More exactly, we verified that there exists a constant C depending only on $\alpha > 0$ such that

$$\frac{1}{\log n \log m} \sum_{\substack{2^{-\alpha} \leq k/l \leq 2^\alpha \\ (k,l) \leq (n,m)}} \frac{\|s_{k,l}f\|_1}{kl} \leq C \|f\|_{H_1(\mathbb{T}^2)}.$$

Note that the space $H_1(\mathbb{T}^2)$ defined in [12] is different from $H_1(\mathbb{T} \times \mathbb{T})$ used here. With the help of Riesz and conjugate transforms one can show that $\|\cdot\|_{H_1(\mathbb{T}^2)} \leq \|\cdot\|_{H_1(\mathbb{T} \times \mathbb{T})}$. We obtained also the convergence result

$$\frac{1}{\log n \log m} \sum_{\substack{2^{-\alpha} \leq k/l \leq 2^\alpha \\ (k,l) \leq (n,m)}} \frac{\|s_{k,l}f - f\|_1}{kl} \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

where $f \in H_1(\mathbb{T}^2)$. The analogous results for two-parameter Walsh-Fourier series can also be found in [12].

In this paper we extend these theorems to the d -dimensional case and prove an even stronger inequality for $f \in H_1(\mathbb{T} \times \cdots \times \mathbb{T})$:

$$\frac{1}{\prod_{i=1}^d \log n_i} \sum_{i=1}^d \sum_{k_i=1}^{n_i} \frac{\|s_k f\|_{H_1(\mathbb{T} \times \cdots \times \mathbb{T})}}{\prod_{i=1}^d k_i} \leq C \|f\|_{H_1(\mathbb{T} \times \cdots \times \mathbb{T})}$$

where C is an absolute constant. From this it follows easily that

$$\lim_{n \rightarrow 0} \frac{1}{\prod_{i=1}^d \log n_i} \sum_{i=1}^d \sum_{k_i=1}^{n_i} \frac{\|s_k f - f\|_{H_1(\mathbb{T} \times \cdots \times \mathbb{T})}}{\prod_{i=1}^d k_i} = 0$$

whenever $f \in H_1(\mathbb{T} \times \cdots \times \mathbb{T})$. We extend these results also to $p < 1$, which was unknown even in the one-parameter case.

In the proof we have to use a different method than in [12], we use the multi-parameter Hardy-Littlewood inequality (see JAWERTH and TORCHINSKY [8]) and the fact that the maximal operator of the Cesàro means of a distribution is bounded from $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ to $L_p(\mathbb{T}^d)$ (see WEISZ [13]).

Moreover, we extend Bernstein's inequality to multi-parameter trigonometric polynomials. We investigate also multiplier operators and give

a sufficient condition for the multiplier such that the operator is bounded on the Hardy space.

I would like to thank the referee for reading the paper carefully and for his useful comments.

2. Hardy spaces and conjugate functions

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^d$ be its Cartesian product taken with itself d -times, moreover, let $\mathbb{T} := [-\pi, \pi)$ and λ be the Lebesgue measure. We briefly write L_p instead of the $L_p(\mathbb{T}^d, \lambda)$ space while the norm (or quasi-norm) of this space is defined by $\|f\|_p := (\int_{\mathbb{T}^d} |f|^p d\lambda)^{1/p}$ ($0 < p \leq \infty$).

For $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $x = (x_1, \dots, x_d) \in \mathbb{T}^d$ set $n \cdot x := \sum_{i=1}^d n_i x_i$. Let f be a distribution on $C^\infty(\mathbb{T}^d)$. The n th Fourier coefficient is defined by $\hat{f}(n) := f(e^{-in \cdot x})$ where $i = \sqrt{-1}$ and $n \in \mathbb{Z}^d$. In the special case when f is an integrable function then

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-in \cdot x} dx.$$

For a distribution f and $z_i := r_i e^{ix_i}$ ($0 < r_i < 1$) let

$$u(z) = u(r_1 e^{ix_1}, \dots, r_d e^{ix_d}) := (f * P_{r_1} \times \cdots \times P_{r_d})(x) \quad (x \in \mathbb{T}^d)$$

where $*$ denotes the convolution and

$$P_r(y) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{iky} = \frac{1-r^2}{1+r^2-2r \cos y} \quad (y \in \mathbb{T})$$

is the Poisson kernel. It is easy to show that $u(z)$ is a multi-harmonic function.

Let $0 < \alpha < 1$ be an arbitrary number. We denote by $\Omega_\alpha(x)$ ($x \in \mathbb{T}$) the region bounded by two tangents to the circle $|z| = \alpha$ from e^{ix} and the longer arc of the circle included between the points of tangency. The non-tangential maximal function is defined by

$$u_\alpha^*(x) := \sup_{z_i \in \Omega_{\alpha_i}(x_i)} |u(z)| \quad (0 < \alpha_i < 1; i = 1, \dots, d).$$

The Hardy space $H_p(\mathbb{T} \times \cdots \times \mathbb{T}) = H_p$ ($0 < p \leq \infty$) consists of all distributions f for which $u_\alpha^* \in L_p$ and set

$$\|f\|_{H_p} := \|u_{1/2, \dots, 1/2}^*\|_p.$$

The equivalence $\|u_\alpha^*\|_p \sim \|f\|_{H_p}$ ($0 < p \leq \infty$, $0 < \alpha_i < 1$) was proved in FEFFERMAN, STEIN [4] and GUNDY, STEIN [7].

For a distribution

$$f \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in \cdot x}$$

the *conjugate distributions* are defined by

$$\tilde{f}^{(j_1, \dots, j_d)} \sim \sum_{n \in \mathbb{Z}^d} \left(\prod_{i=1}^d (-i \operatorname{sign} n_i)^{j_i} \right) \hat{f}(n) e^{in \cdot x} \quad (j_i = 0, 1).$$

Note that $\tilde{f}^{(0, \dots, 0)} := f$. GUNDY and STEIN [6], [7] verified that if $f \in H_p$ ($0 < p < \infty$) then all conjugate distributions are also in H_p and

$$(1) \quad \|f\|_{H_p} = \|\tilde{f}^{(j_1, \dots, j_d)}\|_{H_p} \quad (j_i = 0, 1).$$

Furthermore (see also CHANG and FEFFERMAN [2], FRAZIER [5], DUREN [3]),

$$(2) \quad \|f\|_{H_p} \sim \sum_{i=1}^d \sum_{j_i=0}^1 \|\tilde{f}^{(j_1, \dots, j_d)}\|_p,$$

where $\sim$ denotes the equivalences of the spaces and norms.

For a distribution f with Fourier series

$$f \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in \cdot x} \quad \text{let} \quad Pf \sim \sum_{n \in \mathbb{N}^d} \hat{f}(n) e^{in \cdot x}$$

be the Riesz projection. Then $f \in H_p$ if and only if $Pf \in L_p$ and

$$(3) \quad \|f\|_{H_p} \sim \|Pf\|_p \quad (0 < p < \infty)$$

(see GUNDY and STEIN [6], [7]). Moreover, it is known that $H_p \sim L_p$ ($1 < p < \infty$).

In this paper the constants C are absolute constants and the constants C_p are depending only on p and may denote different constants in different contexts.

JAWERTH and TORCHINSKY [8] proved the following theorem.

Theorem A. For every distribution $f \in H_p$

$$\left(\sum_{i=1}^d \sum_{|n_i|=0}^{\infty} \frac{|\hat{f}(n)|^p}{\prod_{i=1}^d |n_i \vee 1|^{2-p}} \right)^{1/p} \leq C_p \|f\|_{H_p} \quad (0 < p \leq 2).$$

Denote by $s_n f$ the n th partial sum of the Fourier series of a distribution f , namely,

$$s_n f(x) := \sum_{i=1}^d \sum_{k_i=-n_i}^{n_i} \hat{f}(k) e^{ik \cdot x}.$$

For $n \in \mathbb{N}^d$ and a distribution f the *Cesàro mean* of order n of the Fourier series of f is given by

$$\sigma_n f := \frac{1}{\prod_{i=1}^d (n_i + 1)} \sum_{i=1}^d \sum_{k_i=0}^{n_i} s_k f = f * (K_{n_1} \times \cdots \times K_{n_d})$$

where

$$K_m(t) := \sum_{|j|=0}^m \left(1 - \frac{|j|}{m+1} \right) e^{ij \cdot t} \quad (m \in \mathbb{N})$$

is the one-dimensional Fejér kernel of order m . It is shown in ZYG-MUND [14] that $K_m \geq 0$ and

$$(4) \quad \int_{\mathbb{T}} K_m(t) dt = \pi \quad (m \in \mathbb{N}).$$

The following result is due to the author [13].

Theorem B. If $f \in H_p$, then

$$\| \sup_{n \in \mathbb{N}^d} |\sigma_n f| \|_p \leq C_p \|f\|_{H_p} \quad (1/2 < p < \infty).$$

3. Strong convergence results

A sequence $(\lambda_k; k \in \mathbb{Z}^d)$ is said to be a *multiplier* and the *multiplier operator* is defined by

$$M_\lambda f(x) := \sum_{k \in \mathbb{Z}^d} \lambda_k \hat{f}(k) e^{ik \cdot x}.$$

Let $(\lambda_k; k \in \mathbb{Z}^d)$ be an even sequence of real numbers, i.e. $\lambda_{\epsilon_1 k_1, \dots, \epsilon_d k_d} = \lambda_k$ for all $\epsilon_i = -1, 1$ and $k \in \mathbb{Z}^d$. Suppose that there exists $K \in \mathbb{N}^d$ such that $\lambda_k = 0$ if $k_j \geq K_j$ for some $j = 1, \dots, d$. Let

$$\Delta^1 \lambda_k := \sum_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} (-1)^{\epsilon_1 + \dots + \epsilon_d} \lambda_{k_1 + \epsilon_1, \dots, k_d + \epsilon_d}$$

be the first and

$$\Delta^2 \lambda_k := \sum_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} (-1)^{\epsilon_1 + \dots + \epsilon_d} \Delta^1 \lambda_{k_1 + \epsilon_1, \dots, k_d + \epsilon_d}$$

be the second difference of (λ_k) .

Lemma 1. *Suppose that (λ_k) is an even multiplier and there exists $K \in \mathbb{N}^d$ such that $\lambda_k = 0$ if $k_j \geq K_j$ for some $j = 1, \dots, d$. If $\Lambda := \sum_{k \in \mathbb{N}^d} \left(\prod_{i=1}^d (k_i + 1) \right) |\Delta^2 \lambda_k| < \infty$ then*

$$\|M_\lambda f\|_{H_p} \leq C_p \Lambda \|f\|_{H_p} \quad (f \in H_p)$$

for every $1/2 < p < \infty$.

PROOF. Applying Abel rearrangement twice and Theorem B we get that

$$\|M_\lambda f\|_p = \left\| \sum_{k \in \mathbb{N}^d} \left(\prod_{i=1}^d (k_i + 1) \right) \Delta^2 \lambda_k \sigma_k f \right\|_p \leq C_p \Lambda \|f\|_{H_p} \quad (1/2 < p < \infty).$$

This together with (1) implies that

$$\|(M_\lambda f)^{\sim(j_1, \dots, j_d)}\|_p = \|M_\lambda \tilde{f}^{(j_1, \dots, j_d)}\|_p \leq C_p \Lambda \|\tilde{f}^{(j_1, \dots, j_d)}\|_{H_p} = C_p \Lambda \|f\|_{H_p}$$

for $j_i = 0, 1$ and $1/2 < p < \infty$. The equivalence (2) proves now the lemma. $\square$

Let us consider the function

$$v(t) := \begin{cases} 1 & \text{if } |t| < 1 \\ 2 - |t| & \text{if } 1 \leq |t| \leq 2 \\ 0 & \text{if } |t| > 2 \end{cases}$$

and the multiplier operator V_{2N} defined by

$$V_{2N}f(x) := \sum_{k \in \mathbb{N}^d} \left(\prod_{i=1}^d v\left(\frac{k_i}{N_i}\right) \right) \hat{f}(k) e^{ik \cdot x}.$$

Lemma 2. *If $1/2 < p < \infty$ then*

$$\|V_{2N}f\|_{H_p} \leq C_p \|f\|_{H_p} \quad (f \in H_p).$$

PROOF. Let $\lambda_k := \prod_{i=1}^d v(\frac{k_i}{N_i})$. It is easy to see that $\Delta^2 \lambda_k = \prod_{i=1}^d \Delta^2 v(\frac{k_i}{N_i})$ and so we have

$$\sum_{k \in \mathbb{N}^d} \left(\prod_{i=1}^d (k_i + 1) \right) |\Delta^2 \lambda_k| = 3^d$$

which proves the result. $\square$

Now we extend the well known Bernstein's inequality from one- to multi-parameter trigonometric polynomials.

Lemma 3. *Let f be a trigonometric polynomial in the i -th variable of order N_i . If $I \subset \{1, \dots, d\}$, then for every $1 \leq p < \infty$*

$$\left\| \left(\prod_{i \in I} \partial_i \right) f \right\|_p \leq C \left(\prod_{i \in I} N_i \right) \|f\|_p.$$

PROOF. Let us define

$$\phi_{N_i, i \in I}(y) := \prod_{i \in I} (K_{N_i-1}(y_i) (e^{iN_i y_i} + e^{-iN_i y_i})) \quad (y = (y_i, i \in I)).$$

Then by (4), $\|\phi_{N_i, i \in I}\|_1 = C$ and

$$\phi_{N_i, i \in I}(y) = \sum_{i \in I} \sum_{|k_i|=0}^{N_i-1} \left(\prod_{i \in I} \left(1 - \frac{|k_i|}{N_i} \right) \left(e^{i(k_i+N_i)y_i} + e^{i(k_i-N_i)y_i} \right) \right).$$

It is easy to see that $\hat{\phi}_{N_i, i \in I}(k) = \prod_{i \in I} \frac{k_i}{N_i}$ for $-N_i \leq k_i \leq N_i$. Then

$$\phi_{N_i, i \in I} * f = - \frac{(\prod_{i \in I} \partial_i) f}{\prod_{i \in I} N_i}$$

proves the lemma. $\square$

Our main result is the following

Theorem 1. *If $f \in H_p$ and $1/2 < p \leq 1$ then*

$$\sup_{N_i \geq 2} \left(\frac{1}{\prod_{i=1}^d \log N_i} \right)^{[p]} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \frac{\|s_k f\|_{H_p}^p}{\prod_{i=1}^d k_i^{2-p}} \leq C_p \|f\|_{H_p}^p$$

where $[p]$ denotes the integer part of p .

PROOF. To avoid some technical difficulties, we prove the theorem for two parameters, only. By (3), it is enough to show that

$$\sup_{N, M \geq 2} \left(\frac{1}{\log N \log M} \right)^{[p]} \sum_{k=1}^N \sum_{l=1}^M \frac{\|s_{k,l}(Pf)\|_p^p}{(kl)^{2-p}} \leq C_p \|Pf\|_p^p$$

whenever $f \in H_p$ and $1/2 < p \leq 1$.

It is easy to see that

$$\sum_{k=1}^N \sum_{l=1}^M \frac{\|s_{k,l}(Pf)\|_p^p}{(kl)^{2-p}} \leq \sum_{k=1}^{2N} \sum_{l=1}^{2M} \frac{\|s_{k,l}(V_{2N,2M}(Pf))\|_p^p}{(kl)^{2-p}}.$$

For fixed x and y , the (k, l) -th Fourier coefficient of

$$\sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}(V_{2N,2M}(Pf))(x, y) e^{ikt} e^{ilu}$$

is $s_{k,l}(V_{2N,2M}(Pf))(x, y)$. Then we can apply Theorem A and (3) to obtain

$$\begin{aligned} & \sum_{k=1}^{2N} \sum_{l=1}^{2M} \frac{|s_{k,l}(V_{2N,2M}(Pf))(x, y)|^p}{(kl)^{2-p}} \\ & \leq C_p \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}(V_{2N,2M}(Pf))(x, y) e^{ikt} e^{ilu} \right|^p dt du. \end{aligned}$$

Using the notation

$$a_{n,m} := v\left(\frac{n}{N}\right) v\left(\frac{m}{M}\right) \hat{f}(n, m) e^{inx} e^{imy}$$

we have

$$\begin{aligned}
\sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}(V_{2N,2M}(Pf))(x,y) e^{ikt} e^{ilu} &= \sum_{k=1}^{2N} \sum_{l=1}^{2M} \sum_{n=1}^k \sum_{m=1}^l a_{n,m} e^{ikt} e^{ilu} \\
&= \sum_{n=1}^{2N} \sum_{m=1}^{2M} a_{n,m} \frac{e^{i(2N+1)t} - 1}{e^{it} - 1} \frac{e^{i(2M+1)u} - 1}{e^{iu} - 1} \\
&\quad + \sum_{n=1}^{2N} \sum_{m=1}^{2M} a_{n,m} \frac{e^{i(2N+1)t} - 1}{e^{it} - 1} \frac{1 - e^{imu}}{e^{iu} - 1} \\
&\quad + \sum_{n=1}^{2N} \sum_{m=1}^{2M} a_{n,m} \frac{1 - e^{int}}{e^{it} - 1} \frac{e^{i(2M+1)u} - 1}{e^{iu} - 1} \\
&\quad + \sum_{n=1}^{2N} \sum_{m=1}^{2M} a_{n,m} \frac{1 - e^{int}}{e^{it} - 1} \frac{1 - e^{imu}}{e^{iu} - 1} \\
&= (A) + (B) + (C) + (D).
\end{aligned}$$

Recall that for the Dirichlet kernel

$$D_N(t) := \frac{1}{2} e^{-iNt} \frac{e^{i(2N+1)t} - 1}{e^{it} - 1}$$

we have

$$\|D_N\|_1 \sim \log N \quad \text{and} \quad |D_N(t)| \leq \frac{C}{t} \quad (N \in \mathbb{N})$$

(see e.g. TORCHINSKY [11]). Applying this, Lemma 2 and (3) we conclude that

$$\begin{aligned}
\int_{\mathbb{T}^4} |(A)|^p dt du dx dy &= \int_{\mathbb{T}^2} \left| \frac{e^{i(2N+1)t} - 1}{e^{it} - 1} \right|^p \left| \frac{e^{i(2M+1)u} - 1}{e^{iu} - 1} \right|^p dt du \\
&\times \|V_{2N,2M}(Pf)\|_p^p \leq \begin{cases} C \log N \log M \|Pf\|_1 & \text{if } p = 1, \\ C_p \|Pf\|_p^p & \text{if } 1/2 < p < 1. \end{cases}
\end{aligned}$$

For the second term we obtain

$$\begin{aligned} \int_{\mathbb{T}^4} |(B)|^p dt du dx dy &= \int_{\mathbb{T}} \left| \frac{e^{i(2N+1)t} - 1}{e^{it} - 1} \right|^p dt \\ &\times \int_{\mathbb{T}} \left| \frac{1}{e^{iu} - 1} \right|^p \int_{\mathbb{T}^2} |V_{2N,2M}(Pf)(x, y) - V_{2N,2M}(Pf)(x, y + u)|^p dx dy du, \end{aligned}$$

which can be estimated by $C_p \|Pf\|_p^p$ if $1/2 < p < 1$ and, moreover, if $p = 1$ then by

$$\begin{aligned} C \log N \int_{|u| < 1/M} \frac{1}{|u|} \int_{\mathbb{T}^2} \left| \int_0^u \partial_2 V_{2N,2M}(Pf)(x, y + w) dw \right| dx dy du \\ + C \log N \int_{|u| \geq 1/M} \frac{1}{|u|} \|V_{2N,2M}(Pf)\|_1 du =: (B_1) + (B_2). \end{aligned}$$

It is easy to see that $(B_2) \leq C \log N \log M \|Pf\|_1$. By Lemma 3,

$$(B_1) \leq C \log N \|V_{2N,2M}(Pf)\|_1 \leq C \log N \|Pf\|_1.$$

The estimation of (C) is similar. Let us consider (D) .

$$\begin{aligned} \int_{\mathbb{T}^4} |(D)|^p dt du dx dy &= \int_{\mathbb{T}^2} \left| \frac{1}{e^{it} - 1} \right|^p \left| \frac{1}{e^{iu} - 1} \right|^p \int_{\mathbb{T}^2} |V_{2N,2M}(Pf)(x, y) \\ &- V_{2N,2M}(Pf)(x, y + u) - V_{2N,2M}(Pf)(x + t, y) \\ &+ V_{2N,2M}(Pf)(x + t, y + u)|^p dx dy dt du. \end{aligned}$$

This can be estimated by $C_p \|Pf\|_p^p$ if $1/2 < p < 1$. In case $p = 1$ we split the integral with respect to t and u into the integrals over the sets $\{|t| < 1/N, |u| < 1/M\}$, $\{|t| < 1/N, |u| \geq 1/M\}$, $\{|t| \geq 1/N, |u| < 1/M\}$ and $\{|t| \geq 1/N, |u| \geq 1/M\}$ and we denote these integrals by (D_1) , (D_2) , (D_3) and (D_4) , respectively. Applying Bernstein's inequality we obtain

$$\begin{aligned} (D_1) &\leq C \int_{|t| < 1/N} \int_{|u| < 1/M} \frac{1}{|tu|} \\ &\times \int_{\mathbb{T}^2} \left| \int_0^t \int_0^u \partial_1 \partial_2 V_{2N,2M}(Pf)(x + v, y + w) dv dw \right| dx dy dt du \leq C \|Pf\|_1. \end{aligned}$$

Similarly,

$$(D_2) \leq C \int_{|t| < 1/N} \int_{|u| \geq 1/M} \frac{1}{|tu|} \int_{\mathbb{T}^2} \left| \int_0^t \partial_1 V_{2N, 2M}(Pf)(x+v, y) \right. \\ \left. - \partial_1 V_{2N, 2M}(Pf)(x+v, y+u) dv \right| dx dy dt du \leq C \log M \|Pf\|_1.$$

(D_3) can be estimated in the same way. For (D_4) we have simply

$$(D_4) \leq C \log N \log M \|Pf\|_1,$$

which finishes the proof of Theorem 1. $\square$

The set of the trigonometric polynomials is dense in H_p , so by the usual density argument we can easily verify the next consequence (cf. WEISZ [12]).

Corollary 1. *If $f \in H_p$ and $1/2 < p \leq 1$ then*

$$\lim_{N \rightarrow \infty} \left(\frac{1}{\prod_{i=1}^d \log N_i} \right)^{[p]} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \frac{\|s_k f - f\|_{H_p}^p}{\prod_{i=1}^d k_i^{2-p}} = 0.$$

Since $\|\cdot\|_p \leq \|\cdot\|_{H_p}$, we get

$$\lim_{N \rightarrow \infty} \frac{1}{\prod_{i=1}^d \log N_i} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \frac{\|s_k f - f\|_1}{\prod_{i=1}^d k_i} = 0$$

whenever $f \in H_1$, which was proved by SMITH [10] in the one-parameter case.

We now give the dual inequality to Theorem 1, which is a Marcinkiewicz–Zygmund type inequality for the BMO space, where BMO is the dual of H_1 . Since the proof is similar to that of Theorem 3 in WEISZ [12], we omit it.

Theorem 2. *If g^k ($k \in \mathbb{N}^d$) are uniformly bounded in BMO then*

$$\sup_{N_i \geq 2} \left\| \frac{1}{\prod_{i=1}^d \log N_i} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \frac{s_k g^k}{\prod_{i=1}^d k_i} \right\|_{BMO} \leq C \sup_{k \in \mathbb{N}^d} \|g^k\|_{BMO}.$$

Note that the corresponding results for multi-parameter Walsh–Fourier series are still unknown.

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(Received February 14, 2000; file received November 10, 2000)