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On generalized quasi Einstein manifolds

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Abstract. The notions of a generalized quasi Einstein manifold and a manifold of generalized quasi constant curvature are introduced and some properties of such manifolds are obtained.

Introduction

The notion of a quasi Einstein manifold $(QE)_n$ was introduced in a recent paper [1] by the author and R. K. Maity. According to them a non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called *quasi Einstein* if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1}$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X) \quad \forall X, \text{ and } U \text{ is a unit vector field.} \tag{2}$$

This paper deals with *generalized quasi Einstein manifolds* which are defined as follows:

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called a generalized quasi Einstein manifold $G(QE)_n$ if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \tag{3}$$

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where a, b, c are scalars of which $b \neq 0$, A, B are non-zero 1-forms such that

$$g(X,U) = A(X), \qquad g(X,V) = B(X) \quad \forall X \tag{4}$$

and U, V are two unit vector fields perpendicular to each other.

In such a case a, b, c are called the associated scalars, A, B are called the associated 1-forms and U, V are called *generators* of the manifold. Such an n-dimensional manifold shall be denoted by the symbol $G(QE)_n$.

If $c = 0$, then (3) takes the form (1) and the manifold becomes a quasi Einstein manifold introduced by Chaki and Maity. This justifies the name generalized quasi Einstein manifold for the manifold defined by (3) and the use of the symbol $G(QE)_n$ for an n-dimensional manifold of this kind.

In 1972 Chen and Yano [2] introduced the notion of a manifold of quasi-constant curvature. According to them a Riemannian *manifold* (M^n, g) $(n > 3)$ is said to be of *quasi-constant curvature* $(QC)_n$ (see also [3]) if its curvature tensor $'R$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} 'R(X,Y,Z,W) = a\big[&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\ &+ b\big[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)\big] \end{aligned} \tag{5}$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \quad \forall X, \tag{6}$$

U being a unit vector field. In such a case a and b are called the associated scalars, A is called the associated 1-form and U is called the *generator* of the manifold. Such an n-dimensional manifold was denoted by the symbol $(QC)_n$ [3].

A generalization of a manifold of quasi-constant curvature, called a manifold of *generalized quasi-constant curvature* is needed for the study of a $G(QE)_n$. Such a manifold is defined as follows: A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ shall be called a manifold of generalized quasi-constant curvature if its curvature tensor $'R$ of type $(0,4)$ satisfies the condition

$$
\begin{aligned}
(7)\quad {}'R(X,Y,Z,W) = {}& a\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\
&+ b\big[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\
&+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)\big] \\
&+ c\big[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\
&- g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\
&+ g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\
&- g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}\big]
\end{aligned}
$$

where a, b, c are scalars of which $b \neq 0$, A, B are two non-zero 1-forms such that

$$(8)\qquad g(X,U) = A(X), \quad g(X,V) = B(X) \quad \forall X,$$

where U, V are two mutually perpendicular unit vector fields. Such an n-dimensional manifold shall be denoted by the symbol $G(QC)_n$. In such a case a, b, c are called the associated scalars, A, B are called the associated 1-forms and U, V are called the generators of the manifold. If $c = 0$, then (7) takes the form (5) and the manifold becomes a manifold of quasi constant curvature. This justifies the name manifold of generalized quasi constant curvature and the use of the symbol $G(QC)_n$ for an n-dimensional manifold of this kind.

In this paper it is shown that in a $G(QE)_n$ the scalars a and $a+b$ are the Ricci curvatures in the directions of the vector fileds V and U respectively and the scalar c is $< \frac{1}{\sqrt{2}}\ell$, where ℓ is the length of the Ricci tensor S. Other results established in this paper are as follows:

i) Every $G(QE)_3$ is a $G(QC)_3$, but a $G(QE)_n$ $(n > 3)$ is not, in general a $G(QC)_n$.

ii) A $G(QE)_n$ $(n > 3)$ is a $G(QC)_n$ if it is conformally flat.

iii) A $G(QC)_n$ $(n > 3)$ is a conformally flat $G(QE)_n$.

iv) If $U^{\perp}$ denotes the $(n-1)$-dimensional distribution of a $G(QE)_n$ $(n > 3)$ orthogonal to the generator U, then the sectional curvature of the plane determinated by the vectors X, Y is $\frac{(3n-2)a+b}{(n-1)(n-2)}$ when $X, Y \in U^{\perp}$, while the sectional curvature of the plane determined by the vectors X, U is $\frac{(3n-2)a+nb}{(n-1)(n-2)}$ when $X \in U^{\perp}$.

1. Scalar curvature and the associated scalars of a $G(QE)_n$ $(n \geq 3)$

In this section we consider a $G(QE)_n$ with associated scalars a, b, c associated 1-forms A, B and generators U and V corresponding to A and B, respectively. Then (3) and (4) will hold. Since U and V are mutually perpendicular unit vector fields, we have

$$g(U,U) = 1, \quad g(V,V) = 1 \quad \text{and } g(U,V) = 0. \tag{1.1}$$

In virtue of (8) $g(U,V) = 0$ can be expressed as

$$A(V) = B(U) = 0. \tag{1.2}$$

Now contracting (3) over X and Y we get

$$r = na + b \tag{1.3}$$

where r is the scalar curvature. Again from (3) we get

$$S(U,U) = a + b \tag{1.4}$$

$$S(V,V) = a \tag{1.5}$$

$$S(U,V) = c. \tag{1.6}$$

If X is a unit vector filed, then $S(X,X)$ is the Ricci curvature in the direction of X. Hence, from (1.4) and (1.5), we can state that $a+b$ and a are the Ricci curvatures in the directions of U and V, respectively.

Let

$$g(LX,Y) = S(X,Y) \tag{1.7}$$

and ℓ^2 denote the square of the length of the Ricci tensor S. Then

$$\ell^2 = S(Le_i, e_i), \tag{1.8}$$

where $\{e_i\}$, $i = 1, 2, \ldots n$ is an orthonormal basis of the tangent space (T_pM) of $G(QE)_n$. Now from (3) we get

$$S(Le_i, e_i) = (n-1)a^2 + (a+b)^2 + 2c^2.$$

Hence

$$\ell^2 - 2c^2 = (n-1)a^2 + (a+b)^2.$$

Therefore

$$\ell^2 > 2c^2 \quad \left[\because \ell^2 - 2c^2 \neq 0\right]. \tag{1.9}$$

From (1.9) we get (1.10) $c < \frac{1}{\sqrt{2}}\ell$.

Summing up, we can state the following theorem:

Theorem 1. *In a $G(QE)_n$ $(n \geq 3)$ the scalars $a+b$ and a are the Ricci curvatures in the directions of the generators U and V, respectively and the associated scalar c is less than $\frac{1}{\sqrt{2}}\ell$, where ℓ is the length of the Ricci tensor S.*

2. Three-dimensional generalized quasi Einstein manifold

In this section we consider a $G(QE)_3$. In this case (2.1) $r = 3a + b$.

It is known [5] that in a 3-dimensional Riemannian manifold (M^3, g) the curvature tensor $'R$ of type $(0,4)$ has the following form:

$$\begin{aligned} 'R(X,Y,Z,W) = &\big[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+ g(X,W)S(Y,Z) - g(Y,W)S(X,Z)\big] \\ &+ \frac{r}{2}\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W). \end{aligned} \tag{2.1}$$

Using (3), we can express (2.1) as follows:

$$\begin{aligned} 'R(X,Y,Z,W) = &\Big(2a+\frac{r}{2}\Big)\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\ &+b\big[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)\big] \\ &+ c\big[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\ &- g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\ &+ g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ &- g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}\big]. \end{aligned} \tag{2.2}$$

In virtue of (7), it follows from (2.2) that a $G(QE)_3$ is a $G(QC)_3$. This leads to the following result:

Theorem 2. *Every $G(QE)_3$ is a $G(QC)_3$.*

3. Conformally flat $G(QE)_n$ $(n > 3)$

A $G(QE)_n$ $(n > 3)$ is not, in general, a $G(QC)_n$. In this section we consider a conformally flat $G(QE)_n$ $(n > 3)$ and show that such a $G(QE)_n$ is a $G(QC)_n$.

It is known [6] that in a conformally flat Riemannian manifold (M^n, g) $(n > 3)$ the curvature tensor $'R$ of type $(0,4)$ has the following form:

$$\begin{aligned}(3.1)\quad {}'R(X,Y,Z,W) = {} & \frac{1}{n-2}\big[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & + g(X,W)S(Y,Z) - g(Y,W)S(X,Z)\big] \\ & + \frac{r}{(n-1)(n-2)}\big[g(Y,Z)g(X,W) \\ & - g(X,Z)g(Y,W)\big].\end{aligned}$$

Using (3) we can express (3.1) as follows:

$$\begin{aligned}(3.2)\qquad {}'R(X,Y,Z,W) = {} & \frac{2a(n-1)+r}{(n-1)(n-2)} \\ & \times\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\ & + \frac{b}{n-2}\big[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ & + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)\big] \\ & + \frac{c}{n-2}\big[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\ & - g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\ & + g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ & - g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}\big].\end{aligned}$$

In virtue of (7) it follows from (3.2) that the manifold under consideration is a $G(QC)_n$ $(\because b \neq 0)$. Therefore we can state as follows:

Theorem 3. *Every conformally flat $G(QE)_n$ $(n > 3)$ is a $G(QC)_n$.*

We now enquire whether every $G(QC)_n$ $(n \geq 3)$ is a $G(QE)_n$. Contracting (7) over Y and Z we get

$$\begin{aligned}(3.3)\qquad S(X,W) = \big[a(n-1)+b\big]g(X,W) + b(n-2)A(X)A(W)\\ + c(n-2)\big[A(X)B(W)+A(W)B(X)\big].\end{aligned}$$

In virtue of (3) it follows from (3.3) that a $G(QC)_n$ $(n \geq 3)$ is a $G(QE)_n$ $(\because b \neq 0)$.

In a Riemannian manifold (M^n, g) $(n > 3)$ the conformal curvature tensor $'C$ of type $(0,4)$ has the following form:

$$\begin{aligned}(3.4)\quad {'C}(X,Y,Z,W) = {'R}(X,Y,Z,W) - \frac{1}{n-2}\big[g(Y,Z)S(X,W)\\ - g(X,Z)S(Y,W) + g(X,W)S(Y,Z) - g(Y,W)S(X,Z)\big]\\ + \frac{r}{(n-1)(n-2)}\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big].\end{aligned}$$

Using (7) and (3.3) it follows from (3.4) that

$$(3.5)\qquad {'C}(X,Y,Z,W) = 0$$

i.e. the manifold under consideration is conformally flat. Thus we can state the following theorem:

Theorem 4. *Every $G(QC)_n$ $(n \geq 3)$ is a $G(QE)_n$ while every $G(QC)_n$ $(n > 3)$ is a conformally flat $G(QE)_n$.*

4. Sectional curvatures at a point of a conformally flat $G(QE)_n$ $(n > 3)$

Let $U^\perp$ denote the $(n-1)$-dimensional distribution in a conformally flat $G(QE)_n$ $(n > 3)$ orthogonal to U. Then for any $X \in U^\perp$, $g(X,U) = 0$ or, $A(X) = 0$. In this section we shall determine sectional curvature K at the plane determined by the vectors $X, Y \in U^\perp$ or by X, U. Putting $Z = Y$ and $W = X$ in (3.1) we get

$$(4.1)\quad {'R}(X,Y,Y,X) = \frac{2a(n-1)+r}{(n-1)(n-2)}\big[g(X,X)g(Y,Y) - \{g(X,Y)\}^2\big].$$

Then

$$K(X,Y) = \frac{'R(X,Y,Y,X)}{g(X,X)g(Y,Y) - \{g(X,Y)\}^2} \tag{4.2}$$
$$= \frac{2a(n-1)+r}{(n-1)(n-2)} \qquad \text{[by (4.1)]}$$
$$= \frac{2a(n-1)+na+b}{(n-1)(n-2)} = \frac{(3n-2)a+b}{(n-1)(n-2)},$$

and

$$K(X,U) = \frac{'R(X,U,U,X)}{g(X,X)g(U,U) - \{g(X,U)\}^2}. \tag{4.3}$$

But

$$'R(X,U,U,X) = \left[\frac{2a(n-1)+r}{(n-1)(n-2)} + \frac{b}{n-2}\right] g(X,X). \tag{4.4}$$

Hence, from (4.3), we get

$$K(X,U) = \frac{2a(n-1)+r}{(n-1)(n-2)} + \frac{b}{n-2} \tag{4.5}$$
$$= \frac{2a(n-1)+na+b}{(n-1)(n-2)} + \frac{b}{n-2}$$
$$= \frac{(3n-2)a+nb}{(n-1)(n-2)}.$$

Summing up we can state the following theorem.

Theorem 5. *In a conformally flat $G(QE)_n$ $(n > 3)$ the sectional curvature of the plane determined by two vectors $X, Y \in U^{\perp}$ is $\frac{(3n-2)a+b}{(n-1)(n-2)}$, while the sectional curvature of the plane determined by two vectors X, U is $\frac{(3n-2)a+nb}{(n-1)(n-2)}$.*

I conclude by stating as follows:

The importance of a $G(QE)_n$ lies in the fact that such a four-dimensional semi-Riemannian manifold is relevant to the study of a general relativistic fluid space-time admitting heat flux [7], where U is taken as the velocity vector field of the fluid and V is taken as the heat flux vector field.

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