

Homeomorphisms and monotone vector fields

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Abstract. A classical result of MINTY [8] states that for a Hilbert space H and a continuous monotone map $A : H \rightarrow H$ the map $A + I$ is a homeomorphism of H . We extend this result to Hadamard manifolds.

1. Introduction

Let B be a Banach space and G a subset of B . The map $A : G \rightarrow B^*$ is called monotone with respect to duality (or in the sense of Minty–Browder) if $\langle Ay - Ax, y - x \rangle \geq 0$ for any x and y in G , where B^* is the dual of B and $\langle \cdot, \cdot \rangle$ is the natural pairing. If the strict inequality holds whenever $x \neq y$, then A is called strictly monotone. If B is a Hilbert space, then the pairing $\langle \cdot, \cdot \rangle$ can be identified with the scalar product of B . We extended the notion of monotonicity for vector fields of a Riemannian manifold. A classical result of MINTY [8] states that for a Hilbert space H and a continuous monotone map $A : H \rightarrow H$ the map $A + I : H \rightarrow H$, where I is the identical map of H , is a homeomorphism. This result (and different variations of it) is widely used to prove existence and uniqueness theorems for operator equations, partial differential equations and variational inequalities (see [19]). Surprisingly, in the finite dimensional case this result boils down just to the continuity and expansivity of $A + I$, being a particular case (it is not trivial to show) of a classical homeomorphism theorem

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of BROWDER [4, Theorem 4.10] (connected to this subject see also [1]–[3], [6], [13]–[15].) We shall generalize this result for a complete connected Riemannian manifold M . We shall prove that a continuous expansive map $A : M \rightarrow M$ is a homeomorphism. By an expansive map on a Riemannian manifold we mean a map which increases the distance between any two points. The distance function on a Riemannian manifold is given by [5, p. 146, Definition 2.4]. The expansivity of A can be greatly weakened. It is enough to suppose that A is reverse uniform continuous, which means that for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $d(Ax, Ay) < \delta$ implies $d(x, y) < \varepsilon$, where d denotes the distance function on M . Particularly if M is an Hadamard manifold (complete, simply connected Riemannian manifold, of nonpositive sectional curvature) and X is a monotone vector field on M we shall prove that $\exp X$ is expansive. Hence if X is continuous $\exp X$ is a homeomorphism of M , extending Minty's classical result. (We note that for a Hilbert space H we have $\exp X = X + I$, where X is identified with a map of H .)

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2. Preliminary results

First we prove the following lemma:

Lemma 2.1. *Consider $\mathbb{R}^2$ endowed with the canonical scalar product $\langle \cdot, \cdot \rangle$. Denote by $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$. Let $abcd$ be a quadrilateral in $\mathbb{R}^2$ such that $\|c - d\| > \|a - b\|$. Denote by α, β, γ and δ the angles $\angle dab, \angle abc, \angle bcd$ and $\angle cda$, respectively. Then*

$$(2.1) \quad \|a - d\| \cos \delta + \|b - c\| \cos \gamma > 0.$$

(This holds even if $abcd$ degenerates to a triangle.)

PROOF. If $a = b$ the inequality follows from the relation

$$\|a - d\| \cos \delta + \|a - c\| \cos \gamma = \|c - d\|,$$

which can be easily obtained by projecting a to the straight line joining c and d . Suppose that $a \neq b$. From $\|c - d\| > \|a - b\|$ and the Schwarz inequality we have that

$$\langle d - c, a - b \rangle < \|d - c\|^2,$$

which is equivalent to

$$(2.2) \quad \langle c - d, a - d \rangle + \langle d - c, b - c \rangle > 0.$$

It is easy to see that (2.2) implies (2.1). $\square$

In the following definition indices $i = 1, \dots, n$ are considered modulo n . A *geodesic n -sided polygon* in a Riemannian manifold M is a set formed by n segments of minimizing unit speed geodesics (called *sides* of the polygon)

$$\gamma_i : [0, l_i] \rightarrow M; \quad i = 1, \dots, n,$$

in such a way that $\gamma_i(l_i) = \gamma_{i+1}(0)$; $i = 1, \dots, n$. The endpoints of the geodesic segments are called *vertices* of the polygon. The angle

$$\angle(-\dot{\gamma}_i(l_i), \dot{\gamma}_{i+1}(0)); \quad i = 1, \dots, n$$

is called the (interior) *angle* of the corresponding vertex.

Recall that on Hadamard manifolds every two points can be uniquely joined by a geodesic arc [11]. Hence the distance between two points of an Hadamard manifold is the length of the geodesic joining these points.

Let M be an Hadamard manifold. If a, b, c are three arbitrary points of M then ab will denote the distance of a from b and abc_Δ the geodesic triangle of vertices a, b, c (which is uniquely defined). In general a geodesic polygon in M , of consecutive vertices $a_1, \dots, a_n$ will be denoted by $a_1 \dots a_n$.

Lemma 2.2. *Let $abcd$ be a quadrilateral in a Hadamard manifold M and $\alpha, \beta, \gamma, \delta$ the angles of the vertices a, b, c, d , respectively. Then*

$$\alpha + \beta + \gamma + \delta \leq 2\pi.$$

PROOF. Let α_1, α_2 be the angles of the vertex a in adc_Δ and abc_Δ , respectively. Similarly, let γ_1 and γ_2 be the angles of the vertex c in adc_Δ and abc_Δ , respectively. It is known that an angle formed by two edges of

a trieder is bounded by the sum of the other two angles formed by edges.
Hence

$$(2.3) \quad \alpha_1 + \alpha_2 \geq \alpha$$

and

$$(2.4) \quad \gamma_1 + \gamma_2 \geq \gamma.$$

On the other hand by [5, p. 259, Lemma 3.1 (ii)] we have that

$$(2.5) \quad \alpha_1 + \gamma_1 + \delta \leq \pi,$$

$$(2.6) \quad \alpha_2 + \gamma_2 + \beta \leq \pi.$$

Summing inequalities (2.5), (2.6) and using (2.3), (2.4) we obtain

$$\alpha + \beta + \gamma + \delta \leq 2\pi. \quad \square$$

The next lemma follows from [18, Lemma 1].

Lemma 2.3. *Let $(M, \langle \cdot, \cdot \rangle)$ be an Hadamard manifold and $abcd$ be a quadrilateral in M such that α is nonacute and β is obtuse (nonacute), where $\alpha, \beta, \gamma, \delta$ are the angles of the vertices a, b, c, d , respectively. Then $cd > ab$ ($cd \geq ab$).*

The following lemma is a generalization of Lemma 2.1.

Lemma 2.4. *Let $(M, \langle \cdot, \cdot \rangle)$ be an Hadamard manifold and $abcd$ be a quadrilateral in M such that $cd > ab$. Denote by $\alpha, \beta, \gamma, \delta$ the angles of the vertices a, b, c, d , respectively. Then*

$$ad \cos \delta + bc \cos \gamma > 0.$$

(This holds even if $abcd$ degenerates to a triangle.)

PROOF. We identify $T_a M$ with $\mathbb{R}^n$, where $n = \dim M$. Denote by $\|\cdot\|$ the norm generated by the canonical scalar product of $\mathbb{R}^n$.

If $\delta, \gamma \geq \pi/2$ then Lemma 2.3 implies $ab \geq cd$ which contradicts $cd > ab$. Hence we have either $\delta < \pi/2$ or $\gamma < \pi/2$. We can suppose without loss of generality that

$$(2.7) \quad \gamma < \pi/2.$$

The lengths of the sides of a geodesic triangle satisfy the triangle inequalities. Hence there exist the points b', c', d' of $T_a(M)$ such that $\|a - d'\| = ad$, $\|a - c'\| = ac$, $\|d' - c'\| = dc$, $\|a - b'\| = ab$, $\|b' - c'\| = bc$ and b' is contained in the plane of $ad'c'_\Delta$, such that b' and d' are contained in different half planes defined by the straight line in $T_a(M)$ joining a and c' . Let $\alpha' = \angle d'ab'$, $\beta' = \angle ab'c'$, $\gamma' = \angle b'c'd'$, $\delta' = \angle c'd'a$, $\gamma'_1 = \angle ac'd'$ and $\gamma'_2 = \angle ac'b'$. Using Lemma 2.1 to the quadrilateral $ab'c'd'$ we obtain

$$(2.8) \quad \|a - d'\| \cos \delta' + \|b' - c'\| \cos \gamma' > 0.$$

Denote by γ_1, γ_2 the angles of the vertex c in the triangles adc_Δ, abc_Δ , respectively. Then we have, by [5, p. 259, Lemma 3.1 (i)] that

$$(2.9) \quad \delta' \geq \delta.$$

and

$$(2.10) \quad \gamma'_1 + \gamma'_2 \geq \gamma_1 + \gamma_2 \geq \gamma.$$

We consider two cases:

$$1) \gamma'_1 + \gamma'_2 \leq \pi.$$

We have

$$(2.11) \quad \gamma' = \gamma'_1 + \gamma'_2.$$

Relations (2.10) and (2.11) implies

$$(2.12) \quad \gamma' \geq \gamma.$$

Since $\|a - d'\| = ad$, $\|b' - c'\| = bc$ and the cosine function is strictly decreasing on $]0, \pi]$ (2.8), (2.9) and (2.12) imply

$$ad \cos \delta + bc \cos \gamma > 0.$$

$$2) \gamma'_1 + \gamma'_2 > \pi.$$

If $\delta < \pi/2$ then $ad \cos \delta + bc \cos \gamma > 0$ holds trivially, since $\gamma < \pi/2$. We suppose that $\delta \geq \pi/2$. By (2.9) we have that $\delta' \geq \pi/2$. [5, p. 259, Lemma 3.1 (ii)] implies that

$$(2.13) \quad \gamma'_1 \leq \pi/2.$$

We also have

$$(2.14) \quad \gamma'_2 \leq \pi.$$

Hence (2.13) and (2.14) implies

$$(2.15) \quad 2\pi - \gamma' = \gamma'_1 + \gamma'_2 \leq \frac{3\pi}{2}.$$

By (2.7) and (2.15) we have $0 \leq \gamma < \gamma' \leq \pi$. Since the cosine function is strictly decreasing on $[0, \pi]$ we have

$$(2.16) \quad \cos \gamma > \cos \gamma'.$$

Similarly (2.9) implies

$$(2.17) \quad \cos \delta \geq \cos \delta'.$$

By $\|a - d'\| = ad$, $\|b' - c'\| = bc$, (2.8), (2.16) and (2.17) we have

$$ad \cos \delta + bc \cos \gamma > 0. \quad \square$$

3. Monotone vector fields on Riemannian manifolds

Let M be a Riemannian manifold. We recall that a subset K of M is called (*geodesic*) *convex* [12] if for every two points of M there is a geodesic arc joining these points contained in K .

If N is an arbitrary manifold, we shall denote by $\text{Sec}(TN)$ the family of sections of the tangent bundle TN of N . Using this notation, we have the following definition:

Definition 3.1. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold, $K \subset M$ a convex open set and $X \in \text{Sec}(TK)$ a vector field on K . X is called *monotone* [9] if for every $x, y \in K$ and every unit speed geodesic arc $\gamma : [0, l] \rightarrow M$ joining x and y ($\gamma(0) = x$, $\gamma(l) = y$) contained in K , we have that

$$\langle X_x, \dot{\gamma}(0) \rangle \leq \langle X_y, \dot{\gamma}(l) \rangle,$$

where $\dot{\gamma}$ denotes the tangent vector of γ with respect to the arclength.

Let X be monotone. With the previous notations X is called *strictly monotone* [9] if for every distinct x and y

$$\langle X_x, \dot{\gamma}(0) \rangle < \langle X_y, \dot{\gamma}(l) \rangle.$$

Since the length of the tangent vector of an arbitrary parametrized geodesic is constant, the relations of Definition 3.1 can be given for any parametrization of γ . It is also easy to see that X is monotone (strictly monotone), if and only if for every geodesic γ (arbitrarily parametrized) the $v : \tau \mapsto \langle X_{\gamma(\tau)}, \gamma'(\tau) \rangle$ is monotone (strictly monotone), where $\gamma'(\tau)$ is the tangent vector of γ with respect to its parameter τ .

The following example makes connection between monotone vector fields and monotone operators of a Euclidean space, showing that with few modifications the formers are generalizations of the latters:

Example 3.2. Let E be a Euclidean space, $G \subset E$ an open and convex set and $h : G \rightarrow E$ a monotone (strictly monotone) operator. Then, the vector field $X \in \text{Sec}(TG)$; $x \mapsto h(x)_x$, where $h(x)_x$ is the tangent vector in 0 of the curve $t \mapsto x + th(x)$, is monotone (strictly monotone).

The next remark follows easily from Definition 3.1.

Remark 3.3. If M is an Hadamard manifold, $K \subset M$ a convex open set and $X \in \text{Sec}(TK)$ is a vector field on K then X is monotone if and only if for every $x, y \in K$

$$(3.1) \quad \langle X_x, \exp_x^{-1} y \rangle + \langle X_y, \exp_y^{-1} x \rangle \leq 0,$$

where $\exp : TM \rightarrow M$ is the exponential map of M .

Examples for monotone vector fields on Riemannian manifolds can be found in [9], [10]. We also remark that the gradient of every (geodesic) convex function [12] on a Riemannian manifold is monotone (see [16], [17]).

4. Homeomorphisms of Hadamard manifolds

The following proposition is a consequence of Lemma 2.4.

Proposition 4.1. *Let M be an Hadamard manifold and $X \in \text{Sec}(TM)$ a monotone vector field on M . Then the map $A = \exp X : M \rightarrow M$ defined by $Ax = \exp_x X_x$ is expansive.*

PROOF. Suppose that A is not expansive. Hence there exist x and y in M such that $x'y' < xy$, where $x' = Ax$ and $y' = Ay$. Consider the

quadrilateral $xyy'x'$. Denote by the same letters the angles corresponding to the vertices x and y , respectively. Then by Lemma 2.4 we have

$$(4.1) \quad xx' \cos x + yy' \cos y > 0.$$

It is easy to see that (4.1) is equivalent to

$$(4.2) \quad \langle X_x, \exp_x^{-1} y \rangle + \langle X_y, \exp_y^{-1} x \rangle > 0.$$

But by (3.1) inequality (4.2) contradicts the monotonicity of X . Hence A is expansive. $\square$

Definition 4.2. Let M be a Riemannian manifold and d its distance function, which is a metric on M (see [5, p. 146, Proposition 2.5]). $A : M \rightarrow M$ is called *reverse uniform continuous* if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $d(Ax, Ay) < \delta$ implies $d(x, y) < \varepsilon$.

Let $\alpha \geq 1$ and $L > 0$ be two arbitrary positive constants and $A : M \rightarrow M$ such that for any x and y in M to have $d(Ax, Ay) \geq Ld(x, y)^\alpha$. Then A is reverse uniform continuous. If $\alpha = L = 1$ we obtain the set of expansive maps.

Theorem 4.3. *Let M be a complete connected Riemannian manifold and $A : M \rightarrow M$ a continuous and reverse uniform continuous map. Then A is a homeomorphism. Particularly this is true for A continuous and expansive.*

PROOF. Let $n = \dim M$. It is easy to see that the reverse uniform continuity of A implies that it is injective and $A^{-1} : AM \rightarrow M$ is continuous, where $AM = \{Ax : x \in M\}$. Hence $A : M \rightarrow AM$ is a homeomorphism. It remains to show that $AM = M$. Suppose that we have already proved that AM is closed. Since $A : M \rightarrow AM$ is a homeomorphism, by Brouwer's domain invariance theorem, [7, p. 65] AM is open. Since M is connected and AM is an open and closed subset of M we have $AM = M$. Hence if we prove that AM is closed we are done. For this let us consider a sequence $x'_n = Ax_n$ in M convergent to $x' \in M$ and prove that $x' \in AM$ i.e. there is an $x \in M$ such that $x' = Ax$. Since x'_n is convergent it is a Cauchy sequence. It is easy to see that the reverse continuity of A implies that x_n is also a Cauchy sequence. Since M is complete, by Hopf-Rinow theorem for Riemannian manifolds it is complete as a metric space (see

[5, p. 146]). Hence x_n is convergent. Denote by x its limit. Since A is continuous taking the limit in the relation $x'_n = Ax_n$ as $n \rightarrow \infty$ we obtain $x' = Ax$. $\square$

By Proposition 4.1 we have the following extension to Hadamard manifolds of Minty's classical homeomorphism theorem for monotone maps [8, Corollary of Theorem 4].

Corollary 4.4. *Let M be an Hadamard manifold and X be a continuous monotone vector field. Then $\exp X : M \rightarrow M$ is a homeomorphism.*

In [10] we proved that if $p_1, p_2, \dots, p_n$ are projection maps onto closed convex sets of an Hadamard manifold [18] then the vector field

$$X = -\exp^{-1}(p_1 \circ \dots \circ p_n)$$

defined by

$$X_x = -\exp_x^{-1}[(p_1 \circ \dots \circ p_n)(x)]$$

is continuous and monotone. Hence we have the following corollary:

Corollary 4.5. *Let M be an Hadamard manifold and $p_1, p_2, \dots, p_n$ projection maps onto closed convex sets of M . Then $\exp[-\exp^{-1}(p_1 \circ \dots \circ p_n)]$ is a homeomorphism of M onto M .*

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