

Framed $(2M + 3)$ -dimensional manifolds endowed with a vertical cyclic connection structure

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Abstract. Geometrical and structural properties are proved for a class of framed manifolds which are equipped with a vertical cyclic connection structure.

1. Introduction

Framed manifolds and f -structures have been initiated by K. YANO and M. KON and have subsequently been studied intensively, see for example [1], [19], [22], [18]. We recall that if $M(\phi, \Omega, \xi_r, \eta^r, g)$ is a $(2m + q)$ -dimensional manifold of this kind, then the ξ_r , for $(r = 2m + 1, \dots, 2m + q)$, are the Reeb vector fields (in the large sense) of the f -structure, and $\eta^r = \xi_r^\flat$ their corresponding covectors. One has the following structure equations:

$$(1) \quad \phi^2 = -\text{Id} + \sum \eta^r \otimes \xi_r, \quad \phi \xi_r = 0, \quad \eta^r \circ \phi = 0, \quad \eta^s(\xi_r) = \delta_r^s,$$

where ϕ is a (1.1) tensor field. With respect to g , one has the following relation

$$g(\phi Z, Z') + g(Z, \phi Z') = 0, \quad Z, Z' \in \Xi(M),$$

(i.e. ϕ is skew-symmetric with respect to g). The 2-form Ω of rank $2m$ has

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the following properties

$$(2) \quad \Omega(Z, Z') = g(\phi Z, Z'), \quad \Omega^m \wedge \eta^{2m+1} \wedge \dots \wedge \eta^{2m+q} \neq 0,$$

and is called the fundamental form of the framed manifold.

In the present paper we assume that $r \in \{2m+1, 2m+2, 2m+3\}$ and for the indices a and b we have the following range $a, b \in \{1, \dots, 2m\}$. Under these conditions and with reference to [18], we call θ_b^a , θ_a^r , and θ_s^r the horizontal, the transversal, and the vertical connection forms respectively. We will assume here that the θ_a^r vanish and that the θ_s^r are defined by a cyclic permutation of the Reeb covectors η^r , which means that

$$(3) \quad \theta_s^r = f_s \eta^r - f_r \eta^s, \quad \forall \widehat{r, s, t}$$

where the f_r are scalar fields, called the principal scalars on M . In the sequel we will call

$$(4) \quad \eta = f_r \eta^r, \quad \text{and} \quad \eta^\sharp = \sum f_r \xi_r,$$

the principal pfaffian and the principal vector field of M respectively. Further, let $D_p^\top = \{e_a\}$ and $D_p^\perp = \{\xi_r\}$ be the horizontal, respectively vertical, distribution on M .

In a first step, the following properties are proved.

- (i) The manifold M under consideration may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is a $2m$ -dimensional submanifold tangent to $D_p^\top$ and $M^\perp$ is a 3-dimensional submanifold tangent to $D_p^\perp$, and the immersion $x : M^\top \rightarrow M$ is totally geodesic;
- (ii) the Ricci tensor field $\mathcal{R}$ of $M^\perp$ is expressed by

$$\mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \quad Z \in \Xi(M);$$

- (iii) ξ is harmonic and if V is any vertical vector which has the property to be a skew-symmetric Killing vector field having ξ as generative, then V is an exterior concurrent vector field and by Bochner's theorem $g(V, \xi)$ is closed;
- (iv) the principal scalars f_r define an isoparametric system [20];
- (v) the gradients $df_r^\sharp = \text{grad } f_r$, define a commutative group.

In a second step, and making use of E. Cartan's structure equations involving the curvature 2-forms, one finds that the vertical curvature 2-forms Θ_r^s satisfy

$$\Theta_r^s = \left(\left(\|\xi\|^2 - \frac{f_t^2}{2} \right) \eta^r + f_r f_t \eta^t \right) \wedge \eta^s - \left(\left(\|\xi\|^2 - \frac{f_t^2}{2} \right) \eta^s + f_s f_t \eta^t \right) \wedge \eta^r, \\ \forall \widehat{r, s, t},$$

and consequently, following [19] the above equations prove that $M^\perp$ is a conformally flat submanifold of M .

Finally, the structure 2-form Ω of M is presymplectic. Then, if X is any horizontal vector field and bX ($= -i_X \Omega$) means the symplectic isomorphism, and in addition the 1-form bX is ϕ -closed, it follows that Ω is invariant by X . In consequence of this, X is a 2-covariant recurrent vector field, which in the case under consideration is expressed by

$$\nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X, \quad \lambda \in \Lambda^0 M.$$

2. Preliminaries

Let (M, g) be a Riemannian C^∞ -manifold and let ∇ be the covariant differential operator with respect to the metric tensor g . We assume that M is oriented and ∇ is the Levi-Civita connection of g . Let ΓTM be the set of sections of the tangent bundle, and

$${}^b : TM \xrightarrow{b} T^*M \quad \text{and} \quad {}^\sharp : TM \xleftarrow{\sharp} T^*M$$

the isomorphisms defined by g (i.e. b is the index lowering operator, and ${}^\sharp$ is the index raising operator).

Following [14], we denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM),$$

the set of vector valued q -forms ($q < \dim M$), and we write for the covariant derivative operator with respect to ∇

$$(5) \quad d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM).$$

It should be noticed that in general $d^{\nabla^2} = d^{\nabla} \circ d^{\nabla} \neq 0$, unlike $d^2 = d \circ d = 0$. If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M , and is also called the soldering form of M [4]. Since ∇ is symmetric one has that $d^{\nabla}(dp) = 0$.

A vector field Z which satisfies

$$(6) \quad d^{\nabla}(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM), \quad \pi \in \Lambda^1 M,$$

is defined to be an exterior concurrent vector field [16] (see also [13]). The 1-form π in (6) is called the concurrence form and is defined by

$$(7) \quad \pi = \lambda Z^b, \quad \lambda \in \Lambda^0 M.$$

In this case, if $\mathcal{R}$ is the Ricci tensor of ∇ , one has

$$(8) \quad \mathcal{R}(Z, V) = \varepsilon(n-1)\lambda g(Z, V)$$

($\varepsilon = \pm 1$, $V \in \Xi(M)$, $n = \dim M$).

A function $\mathbb{R}^n \rightarrow \mathbb{R}$ is isoparametric [20] if $\|\nabla f\|^2$ and $\operatorname{div}(\nabla f)$ are functions of f ($\nabla f = \operatorname{grad} f$).

Let $\mathcal{O} = \{e_A \mid A = 1, \dots, n\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \operatorname{covect} \{\omega^A\}$ be its associated coframe. Then E. Cartan's structure equations can be written in indexless manner as

$$(9) \quad \nabla e = \theta \otimes e,$$

$$(10) \quad d\omega = -\theta \wedge \omega,$$

$$(11) \quad d\theta = -\theta \wedge \theta + \Theta.$$

In the above equations θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-forms on M).

3. The main theorem

Let $M(\phi, \Omega, \xi_r, \eta^r, g)$ be a $(2m+3)$ -dimensional C^∞ -manifold with soldering form dp and carrying an f -structure ϕ [22], that is a tensor field

of type (1.1) of rank $2m$ which satisfies

$$(12) \quad \phi^3 + \phi = 0,$$

$$(13) \quad \phi^2 = -\text{Id} + \sum \eta^r \otimes \xi_r, \quad \phi \xi_r = 0, \quad \eta^r \circ \phi = 0,$$

$$(14) \quad g(Z, Z') = g(\phi Z, \phi Z') + \sum \eta^r(Z) \eta^r(Z'),$$

where Id is the identity morphism of M .

If in addition the fundamental 2-form Ω of M satisfies

$$(15) \quad \Omega(Z, Z') = g(\phi Z, Z'), \quad \Omega^m \wedge \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3} \neq 0,$$

then M is known [22] to be a framed f -manifold.

With respect to the cobasis $\mathcal{O}^* = \text{covect}\{\omega^a, \eta^r\}$ of $\mathcal{O} = \text{vect}\{e_a, \xi_r\}$ ($1 \leq a \leq 2m$; $2m + 1 \leq r \leq 2m + 3$), the 2-form Ω is expressed by the standard form

$$(16) \quad \Omega = \sum_{i=1}^m \omega^i \wedge \omega^{i^*}, \quad i^* = i + m.$$

Making use of (9) and (13), one finds the known Kaehlerian relations

$$(17) \quad \theta_j^i = \theta_{j^*}^{i^*}, \quad \theta_j^{i^*} = \theta_i^{j^*}.$$

We recall [18] that one may split the tangent space $T_p(M)$ of M at every point $p \in M$ as

$$(18) \quad T_p(M) = D_p^\top \oplus D_p^\perp,$$

where $D_p^\top = \{e_a \mid a \in \{1, \dots, 2m\}\}$ and $D_p^\perp = \{\xi_r\}$ are two complementary orthogonal distributions, called the horizontal and the vertical distribution respectively. As a consequence of this decomposition, one may write the soldering form as

$$(19) \quad dp = dp^\top \oplus dp^\perp,$$

where $dp^\top = dp|_{D^\top}$ and $dp^\perp = dp|_{D^\perp}$. By reference to [18] (see also [12]), the connection forms θ_b^a , θ_s^r , and θ_r^a are called the horizontal, the vertical, and the transversal connection forms respectively. In the present

paper we assume that the θ_r^a vanish and that the vertical connection forms are defined by a cyclic permutation of the Reeb covectors η^r , that is:

$$(20) \quad \theta_s^r = f_s \eta^r - f_r \eta^s, \quad \forall \widehat{r, s, t} \quad (\text{cyclic}).$$

In the above relations, the f_r are scalar fields, called the principal scalars on M , and setting

$$(21) \quad \eta = f_r \eta^r, \quad \eta^\sharp = \xi = \sum f_r \xi_r,$$

η and ξ are called the principal pfaffian and the principal vector field respectively. Taking into account that

$$(22) \quad \theta_r^a = 0,$$

one derives by (10) and (20) that

$$(23) \quad d\eta^r = \eta \wedge \eta^r.$$

This shows that the Reeb covectors are η^r exterior recurrent forms [3]. In addition, exterior differentiation of (23) and taking into account (21), yields

$$(24) \quad df_r = f_r \eta,$$

which expresses that η is an exact form. Since one has that

$$(d\eta^r) \neq 0, \quad \eta^r \wedge d\eta^r = 0,$$

it follows according to a known definition [6] that in the case under discussion the Reeb covectors are of class 2. Let now

$$(25) \quad \varphi^\perp = \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3}$$

and

$$(26) \quad \varphi^\top = \omega^1 \wedge \dots \wedge \omega^{2m}$$

be the simple unit forms which correspond to the distributions $D_p^\perp$ and $D_p^\top$ respectively. Taking the exterior derivative of (25) and (26), and in view of (20) and (22), one derives that

$$(27) \quad d\varphi^\perp = 0$$

and

$$(28) \quad d\varphi^\top = 0.$$

Hence, in terms of well known terminology [9], the above equations show that $\varphi^\perp$ and $\varphi^\top$ are integral invariants of $D_p^\perp$ and $D_p^\top$ respectively. Therefore, by the theorem of Frobenius, we conclude that the manifold M under consideration may be viewed as the local Riemannian product

$$(29) \quad M = M^\top \times M^\perp,$$

where $M^\top$ is a $2m$ -dimensional manifold tangent to $D^\top$ and $M^\perp$ is a 3-dimensional manifold tangent to $D^\perp$ ($= \{\xi_r\}$).

Remark 3.1. As the tangent space $T_p(M)$, the soldering form dp may be split as

$$dp = dp^\top + dp^\perp,$$

where $dp^\top$ and $dp^\perp$ are the horizontal and the vertical components of dp respectively. In the case under discussion, operating on $dp^\top$ and $dp^\perp$ by the exterior covariant derivative operator d^∇ , one finds

$$(30) \quad d^\nabla(dp^\perp) = 0, \quad d^\nabla(dp^\top) = 0,$$

which, since ∇ is the Levi-Civita connection, leads to

$$d^\nabla(dp) = 0.$$

Using (20), (21), and (22), one gets

$$(31) \quad \nabla \xi_r = f_r dp^\perp - \eta^r \otimes \xi,$$

and one derives

$$(32) \quad [\xi_r, \xi_s] = f_s \xi_r - f_r \xi_s.$$

In view of (24), the covariant differential of $[\xi_r, \xi_s]$ can be expressed as

$$(33) \quad \nabla[\xi_r, \xi_s] = \eta \otimes [\xi_r, \xi_s] - [\xi_r, \xi_s]^\flat \otimes \xi,$$

with which one can check Jacobi's identity

$$\sum_{r,s,t} [\xi_r, [\xi_s, \xi_t]] = 0, \quad \forall \widehat{r, s, t}.$$

Next, operating on (21) with ∇ , and using (20) and (21), one derives that

$$(34) \quad \nabla \xi = \|\xi\|^2 dp^\perp;$$

consequently, following a well known definition [2] one may consider ξ as a concurrent vector field on $M^\perp$. This implies [15] (see also [13]) that ξ is an exterior concurrent vector field on $M^\perp$. Since $\|\xi\|^2 = \sum f_r^2$, one gets at once by (24) that

$$(35) \quad d\|\xi\|^2 = 2\|\xi\|^2 \eta.$$

Therefore, since $d^\nabla(dp^\perp) = 0$, operating on (34) by d^∇ yields

$$(36) \quad d^\nabla(\nabla \xi) = \nabla^2 \xi = 2\|\xi\|^2 \eta \wedge dp^\perp.$$

Hence, by reference to [13], the Ricci tensor field $\mathcal{R}$ of $M^\perp$ is expressed by

$$(37) \quad \mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \quad Z \in \Xi(M).$$

Next, by (24) one may write

$$(38) \quad (df_r)^\sharp = f_r \xi_r, \quad (df_r)^\sharp = \text{grad } f_r,$$

and after further elaboration, one derives that

$$(39) \quad [(df_r)^\sharp, (df_s)^\sharp] = 0, \quad \forall \widehat{r, s, t}.$$

Accordingly we may say that the vector fields $(df_r)^\sharp$, $(df_s)^\sharp$, and $(df_t)^\sharp$ define a commutative group.

Next, by (24) one has that

$$\|\text{grad } f_r\|^2 = \|\xi\|^2 f_r^2,$$

and since

$$\text{div } Z = \text{tr } \nabla Z, \quad Z \in \Xi(M),$$

one derives that

$$\text{div grad } f_r = f_r^3 + \|\xi\|^2 f_r^2, \quad \|\xi\|^2 = \sum f_r^2.$$

Hence, noticing that $[\text{grad } f_r, \text{grad } f_r] = 0$ and on behalf of [20], we conclude from the above relations that the scalars f_r define an isoparametric system.

In another perspective, we recall that the star operator $*$ on an oriented n -dimensional Riemannian manifold (M, g) is an isometric bundle isomorphism between ΛT^*M and itself, and maps $\Lambda^q T^*M$ isomorphically to $\Lambda^{n-q} T^*M$ (see also [14]).

Coming back to the case under consideration, one has

$$(40) \quad \Lambda^q T^*M \rightarrow \Lambda^{2m+3-q} TM.$$

With the usual notation, we denote the codifferential of a p -form by $\delta = (-1)^p *^{-1} d*$, where $*^{-1} = (-1)^{n(n-p)}$ (p is the degree of the form, n is the dimension of the manifold, thus $\delta\omega$ is of degree $p - 1$; see also [14]). Then, in the case under consideration, one deduces that

$$(40) \quad d\delta\eta = 0.$$

Since η is a closed pfaffian, there follows at once that

$$(42) \quad \Delta\eta = 0.$$

This shows that η is a harmonic pfaffian (and consequently $\eta^\sharp$ is a harmonic vector field). Finally, consider the immersion $x : M^\top \rightarrow M$. As it is well known, the second quadratic forms l_r associated with x are defined by

$$(43) \quad l_r = -\langle dp^\top, \nabla \xi_r \rangle.$$

Then, by reference to (31), it can be seen that the l_r vanish, and consequently the immersion $x : M^\top \rightarrow M$ is totally geodesic.

Summarizing, we can formulate the following

Theorem 3.1. *Let $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ be a $(2m + 3)$ -dimensional manifold endowed with a vertical cyclic connection structure and with vanishing transversal connection forms. Let η , $\xi (= \eta^\sharp)$, and f_r be the principal pfaffian, the principal vector field, and the principal scalars on M ; and let $D_p^\top$ and $D_p^\perp = \{\xi_r\}$ be the horizontal and the vertical distributions respectively on M .*

Then any such manifold may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is a $2m$ -dimensional presymplectic submanifold tangent to $D_p^\top$ and $M^\perp$ is a 3-dimensional submanifold tangent to $D_p^\perp$.

The following properties are proved.

- (i) *The immersion $x : M^\top \rightarrow M$ is totally geodesic;*
- (ii) *the principal vector field ξ is an exterior concurrent vector field on $M^\perp$, i.e.*

$$\nabla^2 \xi = 2\|\xi\|^2 \eta \wedge dp^\perp,$$

and this implies

$$\mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \quad Z \in \Xi(M),$$

where $\mathcal{R}$ denotes the Ricci tensor field of $M^\perp$;

- (iii) *the principal pfaffian η is harmonic;*
- (iv) *the vector fields $df_r^\sharp$ define a commutative group, and the scalars f_r define an isoparametric system.*

4. Corollaries

Making use of E. Cartan's structure equations, involving the curvature 2-forms (11), one derives by (20), (23), and (24) that the vertical curvature forms Θ_r^s satisfy

$$(44) \quad \begin{aligned} \Theta_r^s = & \left(\left(\|\xi\|^2 - \frac{f_t^2}{2} \right) \eta^r + f_r f_t \eta^t \right) \wedge \eta^s \\ & - \left(\left(\|\xi\|^2 - \frac{f_t^2}{2} \right) \eta^s + f_s f_t \eta^t \right) \wedge \eta^r, \quad \forall \widehat{r, s, t}. \end{aligned}$$

Then, by reference to [19], the above expressions for Θ_r^s affirm that the vertical submanifold $M^\perp$ of M is a conformally flat submanifold of M .

In another perspective, let

$$(45) \quad V = V^r \xi_r, \quad r \in \{2m+1, 2m+2, 2m+3\},$$

be any vertical vector field on $M^\perp$, and assume that V is a skew-symmetric Killing vector field, having ξ as generative [16] (see also [12]), thus

$$(46) \quad \nabla V = V \wedge \xi,$$

where $\wedge$ denotes the wedge product of vector fields

$$V \wedge \xi = \eta \otimes V - V^\flat \otimes \xi.$$

Since by (31) one gets

$$(47) \quad \nabla V = dV^r \otimes \xi_r + g(V, \xi) dp^\perp - V^\flat \otimes \xi,$$

then comparison of (46) and (47) gives

$$(48) \quad dV^\flat = \eta \wedge V^\flat,$$

which by (48) is in agreement by ROSCA's lemma [16], [17] (see also [12]). Moreover, since V is a Killing vector field and the vector field $\xi (= \eta^\sharp)$, is harmonic, one finds by (21) that

$$(49) \quad dg(V, \xi) = 0,$$

and (49) is in agreement with Bochner's theorem [21], and thus yields a confirmation for the correctness of our computations. In addition, by (34) and (46), one calculates that

$$(50) \quad [V, \xi] = g(V, \xi)\xi,$$

and the above equation means that V defines an infinitesimal conformal transformation of ξ . Operating now on (46) by the operator d^∇ and in view of (34), one gets

$$d^\nabla(\nabla V) = \nabla^2 V = \|\xi\|^2 V^\flat \wedge dp^\perp,$$

which shows that V is an exterior concurrent vector field on $M^\perp$ with $\|\xi\|^2$ as concurrent scalar, and by (6) one may write

$$\mathcal{R}(V, Z) = -2\|\xi\|^2 g(V, Z).$$

On the other hand, by (17) and (22), one finds that

$$(51) \quad d\Omega = 0.$$

Since Ω has constant rank, this means that Ω is a presymplectic form on M . We notice that in this case $\text{Ker}(\Omega)$ coincides with the vertical distribution $D_p^\perp = \{\xi_r\}$ of M , which is also called the characteristic distribution of Ω .

Denote now with the usual notation

$$(52) \quad \Omega^\flat : TM \rightarrow T^*M : Z \rightarrow -i_Z \Omega = {}^\flat Z,$$

the symplectic isomorphism defined by Ω [8]. Since Ω is closed, any vector field X with the property that bX is closed, defines an infinitesimal automorphism of Ω , i.e.

$$(53) \quad \mathcal{L}_X \Omega = 0.$$

Assume that X is a horizontal vector field on M , i.e.

$$X = X^a e_a, \quad a \in \{1, \dots, 2m\}.$$

Then, by (52) one has

$$(54) \quad {}^bX = \sum (X^{i^*} \omega^i - X^i \omega^{i^*}), \quad i \in \{1, \dots, m\}, \quad i^* = i + m,$$

and by the structure equations (10) one gets by exterior differentiation of bX

$$(55) \quad d{}^bX = -(dX^{i^*} + X^a \theta_a^{i^*}) \wedge \omega^i - (dX^i + X^a \theta_a^i) \wedge \omega^{i^*}.$$

Hence, in order for bX to be a ϕ -closed form [16], one must write

$$(56) \quad \begin{cases} dX^i + X^a \theta_a^i = -\lambda \omega^{i^*}, \\ dX^{i^*} + X^a \theta_a^{i^*} = \lambda \omega^i, \end{cases}$$

where λ is a scalar. Taking now the covariant differential of the vector field X , one deduces by (56) and the structure equations (9) that

$$(57) \quad \nabla X = \lambda \phi dp.$$

This shows that X is a ϕ -concurrent vector field. Further, operating on the vector valued 1-form ϕdp by the operator d^∇ , one calculates that

$$d^\nabla(\phi dp) = 0,$$

and therefore it follows from (57) that

$$(58) \quad \nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X.$$

Hence, the above equation proves that the vector field X is, according to well known terminology [10], a 2-covariant recurrent vector field with closed recurrence form.

Summarizing, we proved the following

Theorem 4.1. *The vertical submanifold $M^\perp$ of the manifold M under consideration is conformally flat, and the vertical skew-symmetric Killing vector field V is an exterior concurrent vector field which moreover also defines an infinitesimal conformal transformation of the principal vector field ξ . The structure 2-form Ω of M is presymplectic, and if X is any horizontal vector field for which in addition ${}^bX (= -i_X\Omega)$ is ϕ -closed, then Ω is invariant by X , i.e. $\mathcal{L}_X\Omega = 0$; moreover, X also has the following 2 properties:*

a) X is a ϕ -concurrent vector field, i.e.

$$\nabla X = \lambda \phi dp;$$

b) X is a 2-covariant recurrent vector field with closed recurrence form, i.e.

$$\nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X.$$

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